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NOTE

A Random Coincidence Point Theorem

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We prove a random coincidence point theorem for a pair of commuting random operators which extends results due to T. D. Benavides et al. (1996, *Proc. Amer. Math. Soc.* **124,** 831–838), S. Itoh (1979, *J. Math. Anal. Appl.* **67,** 261–273), K. K. Tan and X. Z. Yuan (1997, *Stochastic Anal. Appl.* **15,** 103–123), and H. K. Xu (1990, *Proc. Amer. Math. Soc.* **110,** 495–500). We also derive a common random point theorem.

Key Words: random coincidence point; random fixed point; Opial's condition; Banach space.

1. INTRODUCTION

Random coincidence point theorems are stochastic generalizations of classical coincidence point theorems. The study of random fixed point theorems was initiated by the Prague school of probabilists in the 1950s. But the interest in these problems was enhanced after the publication of the wonderful survey article of Bharucha-Reid [4]. In 1979, Itoh [5] proved several random fixed point theorems for condensing and nonexpansive random operators. Afterwards, various stochastic analogues of the classical Schauder fixed point theorem have been given by Sehgal and Singh [10], Papageorgiou [9], Lin [7], and Xu [16]. Related (but different) problems have been studied by Benavides et al. [3], Shahzad and Khan [11, 12], Shahzad and Latif [13], Shahzad [14], and Tan and Yuan [15]. Recently Beg and Shahzad [1, 2] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators, which are generalizations of commuting random operators. The aim of this

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paper is to prove a random coincidence point theorem for a pair of commuting random operators which extends the work of Benavides et al. [3], Itoh [5], Tan and Yuan [15], and Xu [16]. A common random fixed point theorem is also obtained.

2. PRELIMINARIES

Let X be a complete metric space and (Ω, Σ) a measurable space. Let $CB(X)$ be the family of all nonempty bounded closed subsets of X and let $K(X)$ denote the family of all nonempty compact subsets of X. A mapping $T: \Omega \to CB(X)$ is called measurable if for any open subset C of X,

$$T^{-1}(C) = \{\omega \in \Omega : T(\omega) \cap C \neq \phi\} \in \Sigma.$$

A mapping $\xi: \Omega \to X$ is said to be a measurable selector of a measurable mapping $T: \Omega \to CB(X)$ if ξ is measurable and for any $\omega \in \Omega$, $\xi(\omega) \in T(\omega)$. A mapping $T: \Omega \times X \to CB(X)$ (resp. $f: \Omega \times X \to X$) is called a random operator if for any $x \in X$, $T(\cdot, x)$ (resp. $f(\cdot, x)$) is measurable. A measurable mapping $\xi: \Omega \to X$ is called a random fixed point of a random operator $T: \Omega \times X \to CB(X)$ (resp. $f: \Omega \times X \to X$) if for every $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))$ (resp. $f(\omega, \xi(\omega)) = \xi(\omega)$). A measurable mapping $\xi: \Omega \to X$ is a random coincidence point of random operators $T: \Omega \times X \to CB(X)$ and $f: \Omega \times X \to X$ if for every $\omega \in \Omega$, $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$.

A Banach space X satisfies Opial's condition [8] if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

holds for all $y \neq x$. Every Hilbert space and the spaces l_p $(1 \leq p < \infty)$ satisfy Opial's condition. A subset M of X is said to be star-shaped with respect to $q \in M$ if $\{(1-t)x + tq : 0 \leq t \leq 1\} \subset M$ for each $x \in M$. The star-shaped subsets include the convex subsets as a proper subclass. A multivalued map $T: X \to CB(X)$ is said to be demiclosed if for every sequence $\{x_n\}$ in X and $y_n \in T(x_n)$, $n = 1, 2, \ldots,$ such that $\{x_n\}$ converges weakly to x and $\{y_n\}$ converges strongly to y, we have $x \in X$ and $y \in T(x)$. A mapping $f: X \to X$ is called weakly continuous if $\{x_n\}$ converges weakly to x implies $\{f(x_n)\}$ converges weakly to $f(x)$. Also, a mapping f on a convex set M is called affine if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for all $x, y \in M$ and $0 \leq t \leq 1$. A random operator $f: \Omega \times X \to X$ is called continuous (weakly continuous, etc.) if for every $\omega \in \Omega$, $f(\omega, .)$ is continuous (weakly continuous, etc.). Similarly, a random operator T: $\Omega \times X \to CB(X)$ is called continuous if for any $\omega \in \Omega$, $T(\omega, .)$ is continuous.

3. MAIN RESULTS

In what follows we shall need the following simple version of Theorem 5.1 of Beg and Shahzad [1] (see also [2]).

THEOREM 3.1. *Let (X, d) be a separable complete metric space, T: $\Omega \times X \to CB(X)$ a multivalued random operator, and f: $\Omega \times X \to X$ a continuous random operator such that $T(\omega, X) \subset f(\omega, X)$ for each $\omega \in \Omega$. If f and T commute and for all $x, y \in X$ and all $\omega \in \Omega$, we have*

$$H(T(\omega, x), T(\omega, y)) \leq kd(f(\omega, x), f(\omega, y)),$$

where $k \in (0, 1)$ and H is the Hausdorff metric on $CB(X)$ induced by the metric d, then T and f have a random coincidence point.

THEOREM 3.2. *Let M be a separable weakly compact subset of a Banach space which is star-shaped with respect to $q \in M$, and let f: $\Omega \times M \to M$ be a continuous affine random operator such that $f(\omega, M) = M$ and $f(\omega, q) = q$ for each $\omega \in \Omega$. Let T: $\Omega \times M \to K(M)$ be a multivalued random operator which commutes with f and for all $x, y \in M$ and all $\omega \in \Omega$, we have*

$$H(T(\omega, x), T(\omega, y)) \leq \|f(\omega, x) - f(\omega, y)\|.$$

If one of the following conditions hold: either (a) $(f - T)(\omega, .)$ *is demiclosed at zero for each $\omega \in \Omega$ or* (b) *f is weakly continuous and X satisfies Opial's condition, then T and f have a random coincidence point.*

Proof. Choose a sequence $\{k_n\}$ of real numbers with $0 < k_n < 1$ and $k_n \to 0$ as $n \to \infty$. For each n, consider the random operator T_n: $\Omega \times M \to K(M)$ defined by

$$T_n(\omega, x) = k_n q + (1 - k_n) T(\omega, x).$$

Then,

$$\begin{aligned} H(T_n(\omega, x), T_n(\omega, y)) &= (1 - k_n) H(T(\omega, x), T(\omega, y)) \\ &\leq (1 - k_n) \|f(\omega, x) - f(\omega, y)\| \end{aligned}$$

for each $x, y \in M$ and each $\omega \in \Omega$. Since $T(\omega, M) \subset M = f(\omega, M)$, we have

$$T_n(\omega, M) \subset f(\omega, M)$$

for each $\omega \in \Omega$. Further, each T_n commutes with f, since for any $x \in M$ and $\omega \in \Omega$, we have

$$\begin{aligned} T_n(\omega, f(\omega, x)) &= k_n q + (1 - k_n) T(\omega, f(\omega, x)) \\ &= k_n f(\omega, q) + (1 - k_n) f(\omega, T(\omega, x)) \\ &= f(\omega, \{k_n q + (1 - k_n) T(\omega, x)\}) \\ &= f(\omega, T_n(\omega, x)). \end{aligned}$$

Since the weak topology is Hausdorff and M is weakly compact, it follows that M is strongly closed and is a complete metric space. Thus, by Theorem 3.1, there is a measurable map $\xi_n: \Omega \to M$ such that

$$f(\omega, \xi_n(\omega)) \in T_n(\omega, \xi_n(\omega))$$

for each $\omega \in \Omega$. For each n, define $F_n: \Omega \to WK(M)$ by

$$F_n(\omega) = \mathrm{w} - \mathrm{cl}\{\xi_i(\omega) : i \geq n\},$$

where $WK(M)$ is the family of all nonempty weakly compact subsets of M and w − cl denotes the weak closure. Define $F: \Omega \to WK(M)$ by $F(\omega) = \bigcap_{n=1}^{\infty} F_n(\omega)$. Because M is weakly compact and separable, the weakly topology on M is a metric topology. Then as in Itoh [5, proof of Theorem 2.5] F is w-measurable and has a measurable selector ξ. This ξ is the desired random coincidence point of f and T. Indeed, fix $\omega \in \Omega$ arbitrarily. Then some subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ converges weakly to $\xi(\omega)$. Also there is some $u_m \in T(\omega, \xi_m(\omega))$ such that

$$f(\omega, \xi_m(\omega)) - u_m = k_m\{q - u_m\}.$$

Since M is bounded and $k_m \to 0$, it follows that $f(\omega, \xi_m(\omega)) - u_m \to 0$. Now $y_m = f(\omega, \xi_m(\omega)) - u_m \in (f - T)(\omega, \xi_m(\omega))$ and $y_m \to 0$. If (a) holds then it follows that $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$. If (b) holds, then, by Lemma 3.1 of Latif and Tweddle [6] $(f - T)(\omega, .)$ is demiclosed, and therefore, f and T have a random coincidence point ξ.

Remark 3.1. Theorem 3.2 extends [3, Corollaries 3.1 and 3.2, 5, Theorem 3.4, 15, Theorem 3.4, 16, Theorem 1(ii)].

THEOREM 3.3. *Suppose that M, f, T, and q satisfy all the hypotheses of Theorem 3.2. If for each $\omega \in \Omega$*

$$f(\omega, x) \in T(\omega, x) \qquad \text{implies the existence of } \lim_n f^n(\omega, x),$$

then T and f have a common random fixed point.

Proof. By Theorem 3.2, there is a measurable map $\xi_0: \Omega \to M$ such that

$$f(\omega, \xi_0(\omega)) \in T(\omega, \xi_0(\omega))$$

for each $\omega \in \Omega$. Now

$$\begin{aligned} f^n(\omega, \xi_0(\omega)) &= f^{n-1}(\omega, f(\omega, \xi_0(\omega))) \in f^{n-1}(\omega, T(\omega, \xi_0(\omega))) \\ &= T(\omega, f^{n-1}(\omega, \xi_0(\omega))) \end{aligned}$$

for each $\omega \in \Omega$. Letting $n \to \infty$, we obtain

$$\xi(\omega) \in T(\omega, \xi(\omega)),$$

where $\xi(\omega) = \lim_n f^n(\omega, \xi_0(\omega))$ (the mapping ξ is the pointwise limit of measurable mappings $f^n(.,\xi_0(.))$, hence measurable). Clearly $\xi(\omega) = f(\omega, \xi(\omega))$ for each $\omega \in \Omega$. Hence, for any $\omega \in \Omega$, $\xi(\omega) = f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$.

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