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Further Improved Variable-Entered Karnaugh Map Procedures for Obtaining the Irredundant Forms of an Incompletely-Specified Switching Function

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ABSTRACT. Out of the potpourri of methods available for traditional minimization of switching functions, the method of the Karnaugh map is distinguished as a quick manual method that provides the user with pictorial insight. An advanced version of this map, *viz*., the variable-entered Karnaugh map (*VEKM*) doubles the variable-handling capability of the map and allows its use for "big" Boolean algebras. The present paper offers a novel exposition of the essential features and properties of the *VEKM,* many of which are published for the first time. It also presents a simple and further improved *VEKM* procedure that obtains one of the irredundant disjunctive forms (*IDFs*) of an incompletely specified switching function (*ISSF*). Duality concepts are used to convert the present procedure into a dual one that obtains an irredundant conjunctive form for an *ISSF*. These procedures differ from their predecessors in two respects. First, the present procedures are rather advanced ones equipped with an explicitly stated set of rules that are clearer, though more powerful, than those of the preceding procedures. Second, the present procedures are more precise in handling the contributions of a map entered term, or alterm, and hence are more likely to capture minor details in the intrinsic structure of the *ISSF* under consideration. Therefore, the present procedures, if followed strictly, are more likely to achieve exact minimality, and even if not, the resulting expressions from them are always guaranteed to be almost minimal. Many detailed examples are given to demonstrate the essential features and properties of the map and to illustrate the rules and steps of the new procedures.

1. Introduction

The variable-entered Karnaugh map (*VEKM*) is a powerful manual tool of many pictorial and pedagogical advantages and a variety of medium-sized applications. Out of the potpourri of existing *VEKM* applications, the following can be singled out: design of combinational logic circuits[1-3], design of sequential logic circuits[4], inversion of Boolean functions[5], differentiation of Boolean functions[6] and probability and reliability analysis[7-10]. The *VEKM* has also many prospective applications. These include the use of the *VEKM* as a pedagogical tool in studying techniques of Boolean reasoning. For example, the *VEKM* can serve as a convenient manual aid for solving medium-sized sets of Boolean equations which may involve "big" Boolean algebras other than the 2-element switching algebra[11]. Moreover, the *VEKM* is also useful for implementing several classes of graph algorithms for problems involving medium-sized graphs.

Several procedures using the *VEKM* for the traditional minimization of a switching function exist. These have been reviewed and compared in[2]. In particular, it is noted that the improved procedures offered by Rushdi[2] include those of Muroga[12], Fletcher[1] and Rushdi[5] as special cases. The present paper presents a simple and further improved *VEKM* procedure that obtains one of the irredundant disjunctive forms (*IDFs*) of an incompletely specified switching function (*ISSF*). Duality concepts are used to convert the present procedure into a dual one that obtains an irredundant conjunctive form for an *ISSF*. These procedures can be thought of as useful extensions or improvements of the procedures in[2], and hence bear many similarities to them. However, the present procedures differ from their predecessors in two respects. First, the present procedures are rather advanced ones equipped with an explicitly stated set of rules that are clearer, though more powerful, than those of the preceding procedures. Second, the present procedures are more precise in handling the contributions of entered terms or alterms, and are more likely to capture minor details in the intrinsic structure of the *ISSF* under consideration. Therefore, the present procedures, if followed strictly, are more likely to achieve exact minimality, and if not, the resulting expressions are always guaranteed to be almost minimal, *i.e.,* to differ only slightly from true minimal expressions.

The rest of the paper is organized as follows. Section 2 discusses the Boole-Shannon Expansion about a single variable and about several variables and utilizes this expansion in constructing *VEKM* representations for completely-specified switching functions. Section 3 presents various *VEKM* representations and demonstrates their interrelationships. The algebraic and *VEKM* representations of incompletely specified switching functions (*ISSFs*) are introduced in section 4. Duality concepts as applied to *VEKM* representations are introduced in section 5. Section 6 presents a simple and further improved *VEKM* procedure for obtaining an *IDF* of an *ISSF* that is guaranteed to be minimal or nearly minimal. Several examples are given in section 7 to illustrate this procedure and the dual procedure that obtains an *ICF* of an *ISSF*. The paper is concluded in section 8 with a discussion of possible work.

2. Boole-Shannon Expansion

The Boole-Shannon expansion is sometimes called "the fundamental theorem of Boolean algebra". It was first discussed by Boole but somehow it is frequently attributed to Shannon[11]. In the following, we discuss the case when this expansion is made about one variable. Later, this case is generalized to allow for several expansion variables.

2.1 Expansion about a Single Variable

***Theorem 1*:** If $f = \{0, 1\}n \rightarrow \{0,1\}$ is a switching function of n variables, then it can be expanded about one of its n variables X_i, $1 \leq i \leq n$, in the form

$$f(\boldsymbol{X}) = \overline{X}_i\, f(\boldsymbol{X} \mid 0_i) \vee X_i F(\boldsymbol{X} \mid 1_i)\,, \tag{1}$$

where $\boldsymbol{X}$ is a vector of n variables

$$\boldsymbol{X} = (X_1, X_2, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, (X_n)\,, \tag{2}$$

and

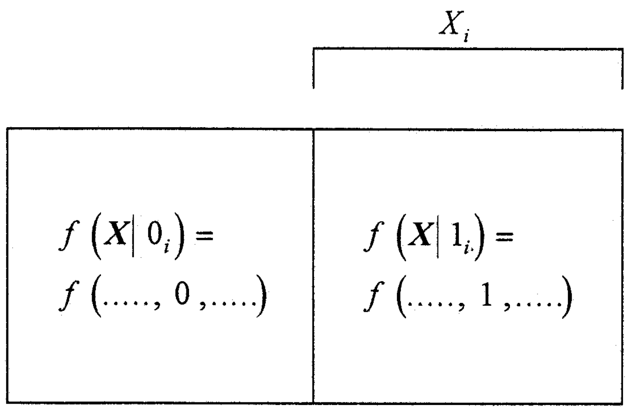
$$f(\boldsymbol{X} \mid 0_i) = f(X_1, X_2, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n)\,, \tag{3}$$

$$f(\boldsymbol{X} \mid 1_i) = f(X_1, X_2, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)\,. \tag{4}$$

Note that $f(\boldsymbol{X} \mid 0_i)$ and $f(\boldsymbol{X} \mid 1_i)$ are functions of the $(n-1)$ remaining variables (other than the expansion variable X_i) and are called subfunctions, restrictions or residuals of the original function. Sometimes we may write these subfunctions simply as $f(\ldots, 0, \ldots)$, and $f(\ldots, 1, \ldots)$, the arguments $\boldsymbol{X} \mid X_i = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ being understood.

***Proof*:** For $X_i = 0$ and $X_i = 1$, the L.H.S. and the R.H.S. of (1) are identical functions in $\boldsymbol{X} \mid X_i$ (Q.E.D.)

Figure 1 gives a pictorial or map representation of (1). The map in Figure 1 is a 2-cell variable-entered Karnaugh map (*VEKM*) of a single map variable. The map entries are the subfunctions of the original function. By contrast, the conventional Karnaugh map (*CKM*) for the same function consists of 2^n cells with constant entries. The subfunctions in the *VEKM* are algebraic expressions or formulas corresponding to 2^{n-1} cells in (or exactly one half of) the corresponding *CKM.* On the other hand, an algebraic expression can be viewed as a *VEKM* of zero map variables.



$$f(\boldsymbol{X}) = f(X_1, X_2, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_n)\,, \qquad 1 \le i \le n$$

FIG. 1. The map representation of the expansion about a single variable.

2.2 Expansion about Several Variables

***Theorem 2*:** If $f = \{0, 1\}^n \to \{0, 1\}$ is a switching function of n variables, then for $0 \le m \le n,$ it can be be expanded about m of its n variables in the form

$$f(X_1, X_2, \ldots, X_{m-1}, X_m, \boldsymbol{Y}) = \bar{X}_1 \bar{X}_2 \ldots \bar{X}_{m-1} \bar{X}_{m-1} \bar{X}_m f_0 \vee$$
$$\bar{X}_1 \bar{X}_2 \ldots \bar{X}_{m-1} X_m f_1 \vee \ldots \vee X_1 X_2 \ldots X_{m-1} X_m f_{(2^m-1)}\,, \tag{5}$$

where

$$\boldsymbol{Y} = [X_{m+1}, X_{m+2}, \cdots, X_n]$$
$$= \text{an } (n-m) - \text{tuple of remaining variables.} \tag{6}$$

$$f_0 = f(0, 0, \ldots, 0, 0, \boldsymbol{Y}), f_1 = f(0, 0, \ldots, 0, 1, \boldsymbol{Y}), \ldots, f_{(2^m-1)}$$
$$= f(1, 1, \ldots, 1, 1, \boldsymbol{Y})\,. \tag{7}$$

***Proof*:** The expansion (5) can be proved by using a truth table of 2^m lines exhibiting the 2^m possible patterns of the m expansion variables $X_1, X_2, \ldots, X_m$. The proof can be viewed as one of "perfect induction" over a reduced set of input combinations of 2^m lines only, with the expansion treated as an "identity" in the remaining variables. Details of the proof are given in Figure 2. (Q.E.D.)

X_1 $X_2 \dots X_{m-1}$ X_m	L.H.S.	R.H.S.
0 0 ... 0 0	$f(0,0,\dots,0,0,\boldsymbol{Y}) = f_0$	$\overline{0}\overline{0}\dots\overline{0}\overline{0}f_0 \vee \overline{0}\overline{0}\dots\overline{0}0f_1 \vee \dots\dots \vee 00\dots00f_{(2^m-1)} = f_0$
0 0 ... 0 1	$f(0,0,\dots,0,1,\boldsymbol{Y}) = f_1$	$\overline{0}\overline{0}\dots\overline{0}\overline{1}f_0 \vee \overline{0}\overline{0}\dots\overline{0}1f_1 \vee \dots\dots \vee 00\dots01f_{(2^m-1)} = f_1$
.	.	.
.	.	.
.	.	.
1 1 ... 1 1	$f(1,1,\dots,1,1,\boldsymbol{Y}) = f_{(2^m-1)}$	$\overline{1}\overline{1}\dots\overline{1}\overline{1}f_0 \vee \overline{1}\overline{1}\dots\overline{1}1f_1 \vee \dots\dots \dots \vee 11\dots11f_{(2^m-1)} = f_{(2^m-1)}$

FIG. 2. Proof of the Boole-Shannon expansion.

When this expansion is associated with a *VEKM* representation, the m expansion variables X_1 , X_2 , ... , X_m are called keystone or map variables while the $e = (n - m)$ remaining variables are called entered variables.

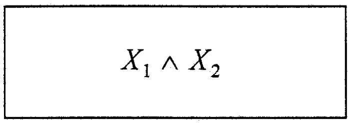
***Theorem 3*:** If $f = \{0, 1\}^n \to \{0, 1\}$ is a switching function of n variables, then it has a total of N = 2^n different *VEKM* representations, each of which corresponds to a distinct Boole-Shannon expansion.

***Proof*:** The No. of possible *VEKM's* = $N = \sum_{m=0}^{n} \binom{n}{m} = (1+1)^n = 2^n$

= the No. of possible Boole-Shannon expansions (Q.E.D.)

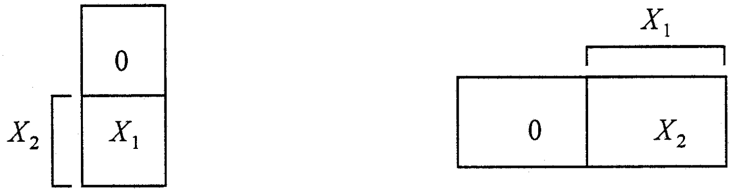
Example 1

The binary AND function $f(X_1 , X_2) = (X_1 \wedge X_2$ has $n =$ and hence 4 *veals,* including the 2 extremes of an algebraic expression ($m = 0$) and a *CKM* ($m = n$), and also including the 2 cases arising from expansion about a single variable ($m = 1$). Figure 3 displays these 4 *veals.*



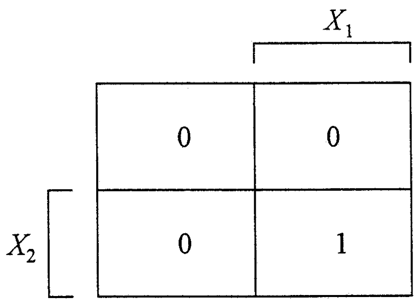
$f(X_1, X_2) = X_1 \wedge X_2$

(a) Algebraic Expression
(*VEKM* with $m = 0$, $e = 2$)



$f(X_1, X_2) = \overline{X}_2(0) \vee X_2(X_1)$ $\quad$ $f(X_1, X_2) = \overline{X}_1(0) \vee X_1(X_2)$

(b) Two *VEKMs* with $m = 1$ and $e = 1$.



$f(X_1, X_2) = \overline{X}_1\overline{X}_2(0) \vee \overline{X}_1 X_2(0) \vee X_1\overline{X}_2(0) \vee X_1 X_2(1)$

(c) *CKM* (*VEKM* with $m = 2$ and $e = 0$).

FIG. 3. Representation of an AND function by 4 different *VEKMs*.

3. Map Representations and their Interrelationships

The purpose of this section is to explain and demonstrate procedures for representing a given switching function of n variables $f(X_1, X_2, \ldots, MN)$ via a *VEKM* of m map variables, where $0 \leq m \leq n$. Without loss of generality, the variables X_1, $X_2, \ldots, MN$ of the function f are assumed to be arranged such that the first m variables among them are the map variables. Usually, we choose these m variables as the most frequently used variables, *i.e.,* as the variables that appear most in a typical algebraic expression of the function. Other choices are also possible[13]. The following example presents various *VEKM* representations for a 4-variable completely-specified switching function and explains their interrelationships.

Example 2

Consider the function $f(A, B, C, D)$ given by the *VEKM* in Figure 4 where A and C are the map or keystone variables and B and D are the entered variables. If the subfunctions f (entries of the *VEKM*) are given by

$$f_0 = B \vee D, f_1 = B \odot D, \; F_2 = B \oplus D, \; f_3 = B \wedge D, \tag{8}$$

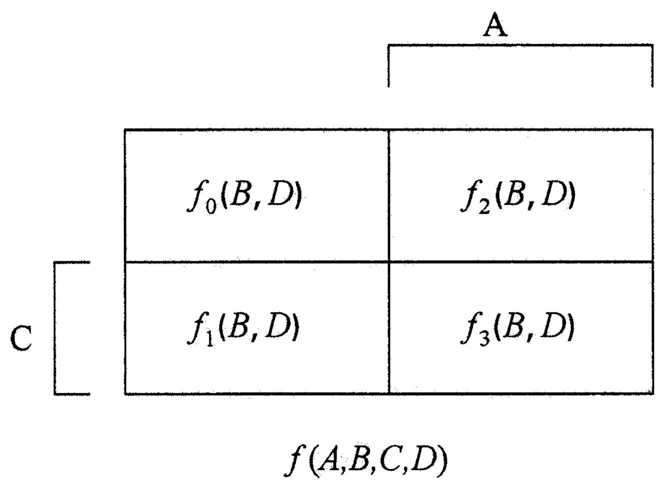


FIG. 4. A *VEKM* representing $f(A,B,C,D)$.

then f is defined by the expression

$$f(A, B, C, D) = \bar{A}\,\bar{C} f_0(B, D) \vee \bar{A}\, C f_1(B, D) \vee$$
$$A\,\bar{C} f_2(B, D) \vee A\, C f_3(B, D)\,. \tag{9}$$

If further, the f_i' are given by their minterm expansions

$$f_0 = \overline{B}\,D \vee B\,\overline{D} \vee B\,D\,,\; F_1 = \overline{B}\,\overline{D} \vee B\,D\,,$$
$$f_2 = B\,\overline{D} \vee \overline{B}\,D,\; f_3 = B\,D, \tag{10}$$

then the expressions (9) of f reduces to its minterm expansion

$$\begin{aligned} f(A, B, C, D) = {} & \overline{A}\,\overline{C}\,(\overline{B}\,D \vee B\,\overline{D} \vee B\,D) \\ & \vee \overline{A}\,C\,(\overline{B}\,\overline{D} \vee B\,D) \\ & \vee A\,\overline{C}\,(\overline{B}\,D\,/B\,\overline{D}) \\ & \vee A\,C\,(B\,D)\,, \end{aligned} \tag{11}$$

and the *VEKM* of Figure 4 can be redrawn as that of Figure 5. Now, consider rearranging the minterm expansion of (11) in terms of the minterms in B and D, namely

$$\begin{aligned} f(A, B, C, D) = {} & \overline{B}\,\overline{D}\,(\overline{A}\,C) \\ & \vee \overline{B}\,D\,(\overline{A}\,\overline{C}\,/\,A\,\overline{C}) \\ & \vee B\,\overline{D}\,(\overline{A}\,\overline{C}\,/A\,\overline{C}) \\ & \vee B\,D\,(\overline{A}\,\overline{C}\,/A\,C)\;. \end{aligned} \tag{12}$$

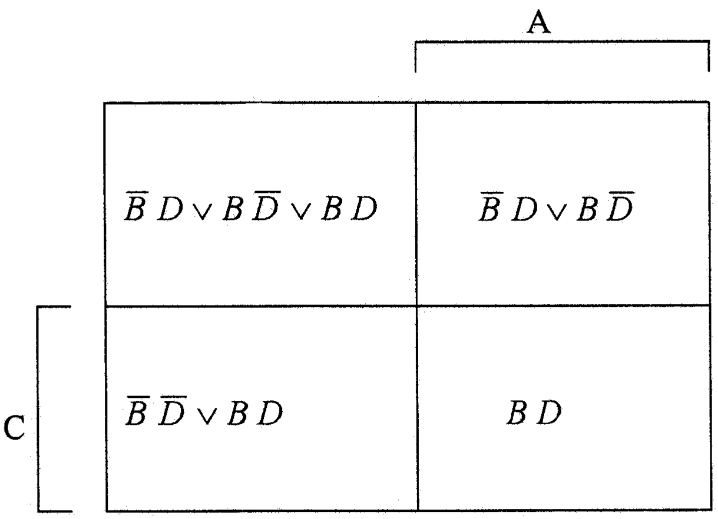


FIG. 5. The same *VEKM* of Figure 4 after the substitution for the values of f_0, f_1, f_2, and f_3.

The *VEKM* in Figure 5 can produce (12) directly by considering that each of the entered terms P_r is ANDed with some associated contribution $Co(P_r)$ as follows

$$f(A, B, C, D) = \bar{B}\,\bar{D}\ Co\,(\bar{B}\,\bar{D}) \vee \bar{B}\,D\ Co\,(\bar{B}\,D) \\ \vee\ B\,\bar{D}\,Co\,(B\,\bar{D}) \vee B\,D\,Co\,(B\,D)\ , \tag{13}$$

where each of the contributions $Co\,(P_r)$ is a minterm expansion in A and C indicating the cell(s) in which the pertinent term P_r appears (Figure 6). Finally, we arrive at the following rule: To obtain a minterm expansion of a completely specified switching function *f,* expressed in *VEKM* form, we express each subfunction/residue/restriction as a minterm expansion of entered variables and then obtain the various contributions of the entered terms as minterm expansions of map variables.

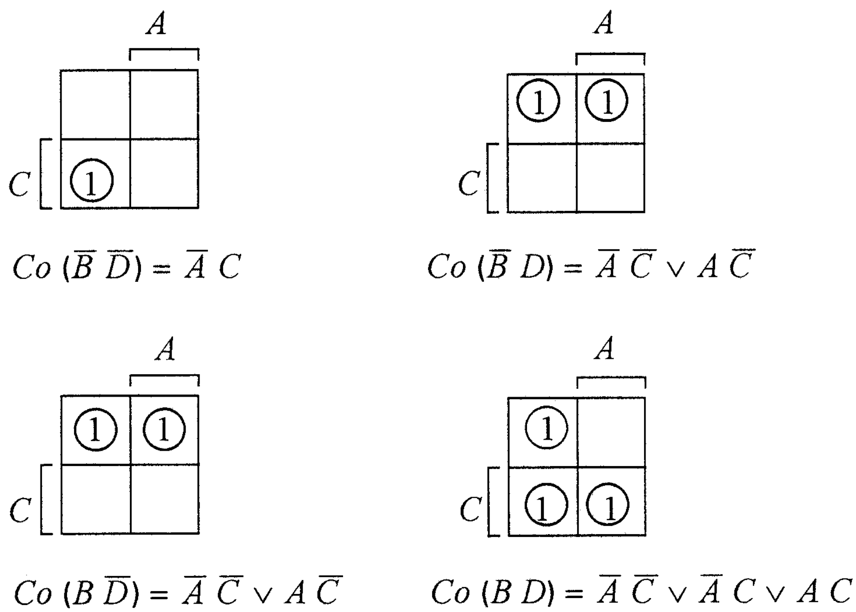


FIG. 6. The *CKMs* representing the contributions of the implicants of the subfunctions shown in Figure 5.

Now, we turn our attention to exploring other *VEKM* representations of our example function given by either of equations (9), (11) or (12) or by Figure 4 or Figure 5. We may increase the number of map variables at the expense of entered variables as follows. Since each entry in Figure 5 is an algebraic expression of B and D (*i.e.* a *VEKM* of 0 map variables), it may be replaced by a *VEKM* of 1 or 2 map variables. Figure 7 shows all subfunctions of Figure 5 replaced by *VEKMs* of a single map variable B and a single entered variable D. Figure 8 is a rearrangement of Figure 7 representing our function by a *VEKM* of 3 map variables, with the single variable D being the sole entered variable. Note that the internal *VEKMs* of Figure 7 have the asserted domain of the map variable B located in such a way as to facilitate the combination of taking place at Figure 8.

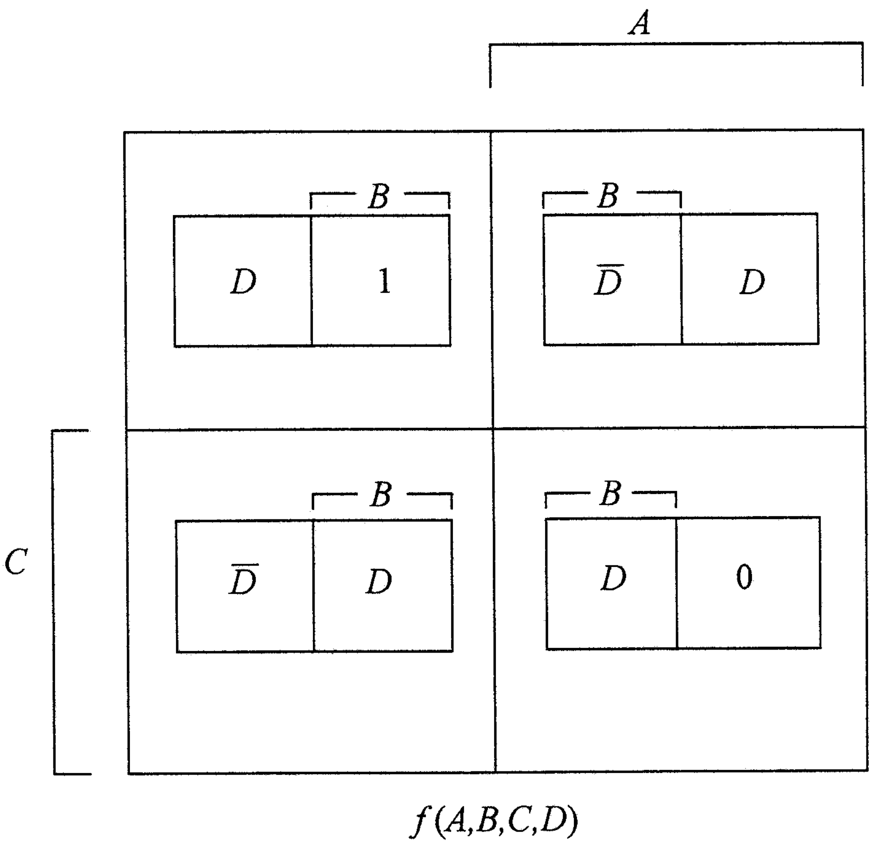


FIG. 7. The *VEKM* of Figure 5 with its subfunctions replaced by *VEKMs* of a single map variable B.

Starting with either of the *VEKM* representations in Figure 5 or in Figure 8 we can similarly produce Figures 9 and 10 and subsequently obtain a *VEKM* of 4 map variables (*i.e.,* a *CKM*) in Figure 11.

4. Incompletely Specified Functions

Many canonical representations of a switching function SF (*e.g.,* the truth table, the K-map, etc.) are equivalent to a listing of the values of the function for the 2^n possible combinations of its n input variables. So far, we have assumed that the truth values were strictly specified for all the 2^n possible input combina-

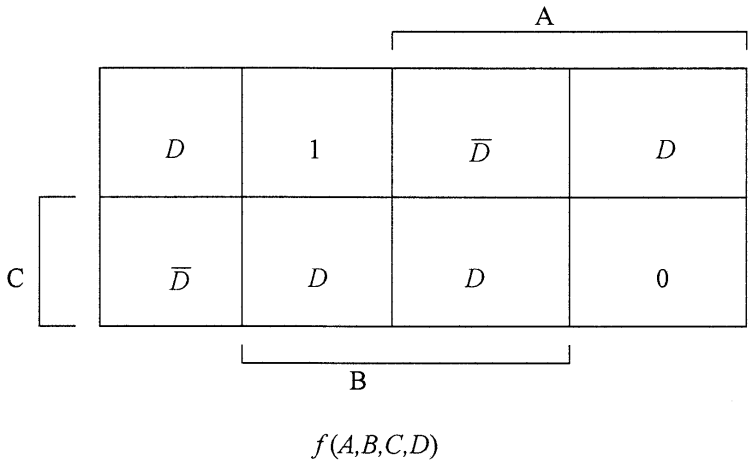


FIG. 8. The function of Figure 5 represented by a *VEKM* of 3 map variables.

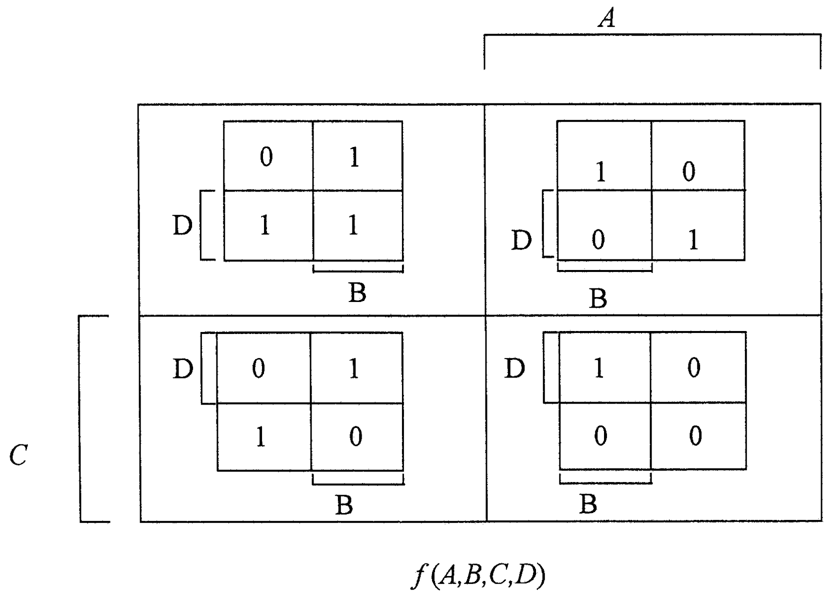


FIG. 9. The *VEKM* of Figure 5 with its subfunctions replaced by *VEKMs* of 2 map variables.

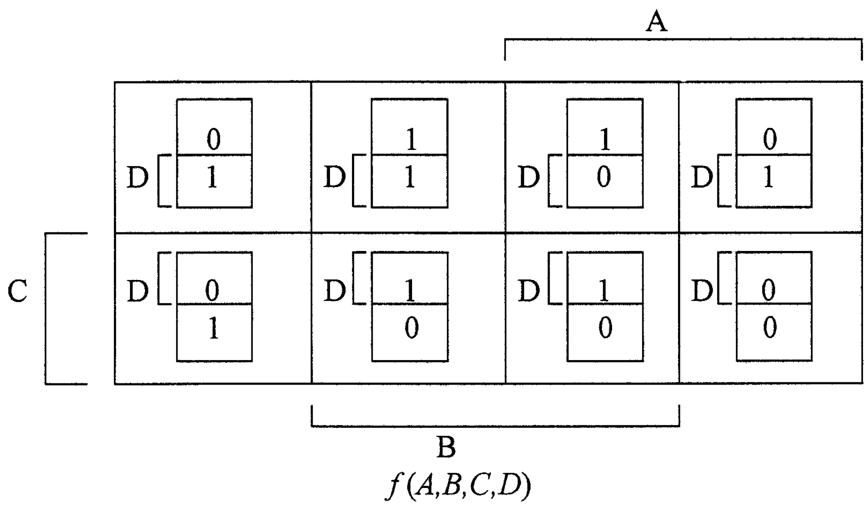


FIG. 10. The *VEKM* of Figure 8 with its subfunctions replaced by *VEKMs* of a single map variable.

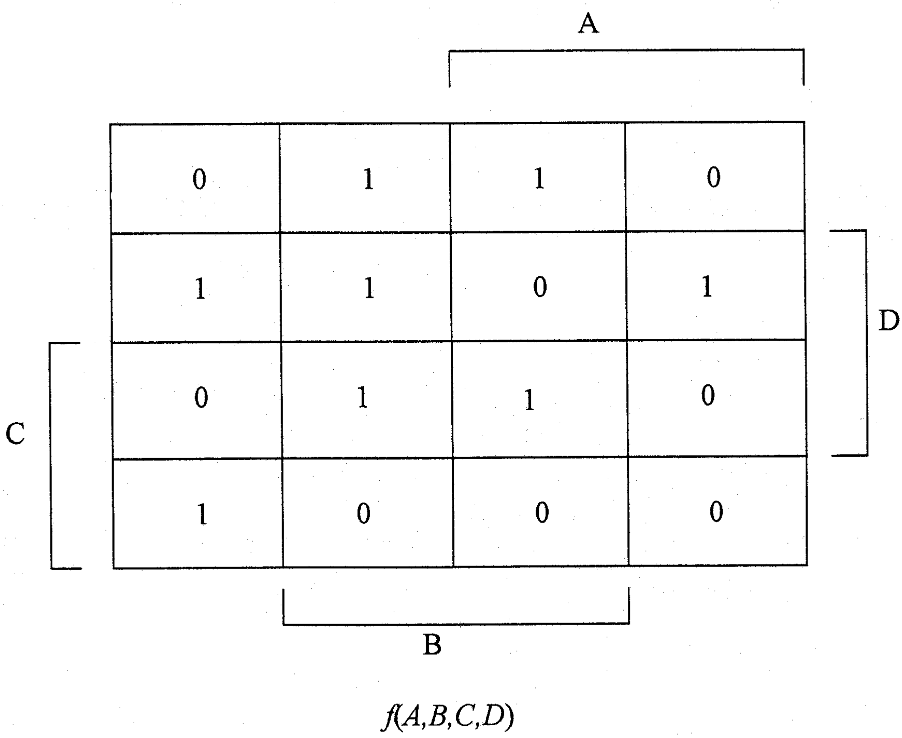


FIG. 11. Rearrangement of the *VEKMs* in Figure 9 and 10 as a *CKM*.

tions. This is not always the case. For example, a logic designer might be designing a switching subcircuit that is a part of a larger circuit in which certain subcircuit inputs occur only under circumstances such that the output of the subcircuit will not influence the overall circuit for these inputs. We obviously *don't care* where the output of the subcircuit in such a case is a 0 or a 1, since it has no effect on the overall circuit.

Another possibility is due to forbidden or don't happen inputs for the circuit under design itself, *viz.,* certain input combinations never occur due to various external constraints. Note that this does not mean that the circuit would not develop some output if a forbidden input occurred. Any switching circuit will respond in some way to any input. However, since the input will never occur, we *don't care* whether the final circuit responds with a 0 or a 1 output to this forbidden- input combination.

When such situations occur, we say that the output is *unspecified.* This is indicated on the truth table by entering d as the functional value instead of 0 or 1. Such conditions are commonly referred to as *don't cares.* A realization of an incompletely specified function is any circuit that produces the same outputs for *all input combinations for which the output is specified*[14].

If an incompletely specified *SF* (*ISSF*) is specified for $L < N$ values in its input domain, then for $(N - L) > 0$ input-domain combinations the value of the function is said to be unspecified, variable {0/1} or a don't care. Therefore, an *ISSF* is not a single function but a set or an interval of $2^{(N-L)}$ *SFs.*

Algebraically, an *ISSF* f can be defined through a pair of completely specified *SFs* (g, h) such that

$$\{ g = 1 \} \Rightarrow \{ f = 1 \} , \tag{14}$$

$$\{ (g = 0) \ \vee \ (h = 0) \} \wedge \ \{ f = 0 \} . \tag{15}$$

Note that f is not specified for the case $\{(g = 0) \wedge (h = 1)\}$. Usually f is written in the form

$$f = g \vee d(h) = g \vee (d \wedge h) , \tag{16}$$

where

$$1 \vee d(h) = 1 , \tag{17a}$$

$$0 \vee d(0) = 0 , \tag{17b}$$

$$0 \vee d(1) = d . \tag{17c}$$

The information contained in (16) and (17) is pictorially conveyed by the *CKM* in Figure 12. This *CKM* is non-standard in one aspect, namely, that its input variables g and h are not necessarily independent, a condition that is usually met (implicitly) by input variables of Karnaugh maps.

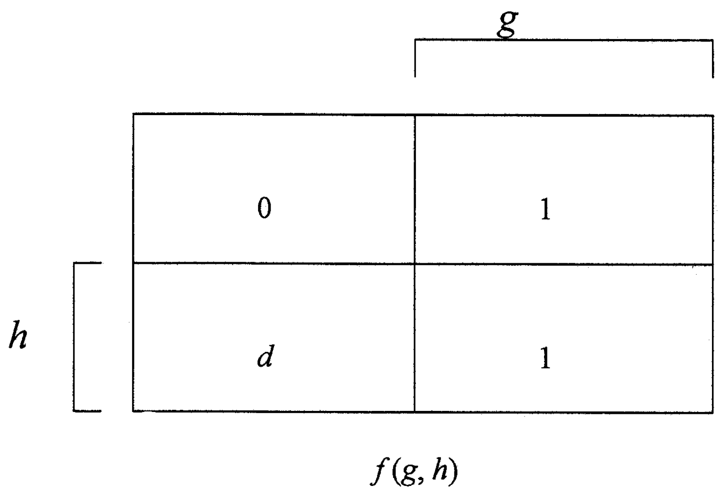


FIG. 12. The *CKM* representing the *ISSF* defined through a pair of *SFs* (g, h).

Example 3

Consider the 4-variable *ISSF* $f(A, B, C, D) = g \vee d(h)$ where

$$g = A\bar{B}C \vee \bar{B}C\bar{D} \vee B\bar{C}D \vee \bar{A}B\bar{C}\,, \tag{18}$$

$$h = \bar{A}\bar{C} \vee \bar{A}\bar{D} \vee ABD\,. \tag{19}$$

The functions g and h are given by the *CKMs* of Figure 13. The function $f = g \vee g \vee d(h)$ can be represented by the *CKM* of Figure 14 which can be obtained by a cellwise combination of the *CKMs* of Figure 13 according to rules (17). Note that the *CKM* of Figure 14 contains 4 different d-cells. These 4 *d's* can be specified independently of one another, so that the *ISSF* of Figure 14 can be reduced to any of $2^4 = 16$ *CSSFs*. It is with this understanding that we should view the symbol d in (16), treating it as an abbreviation of possibly several *d's* and not necessarily a single one.

In retrospect, we note that the function h need not be defined as a completely specified one. A function of the form h' as given by Figure 15 can replace h for all purposes. Note that the *CSSF* h is a special choice of the *ISSF* h'. However, we may not use the *ISSF* h' in our initial definition of *ISSFs* to avoid having a circular definition.

Figure 16 gives an expansion tree for our present *ISSF* f. Starting at the level $m = 0$ (parent node) with f given purely algebraically as $f = g \vee d(h)$, we suc-

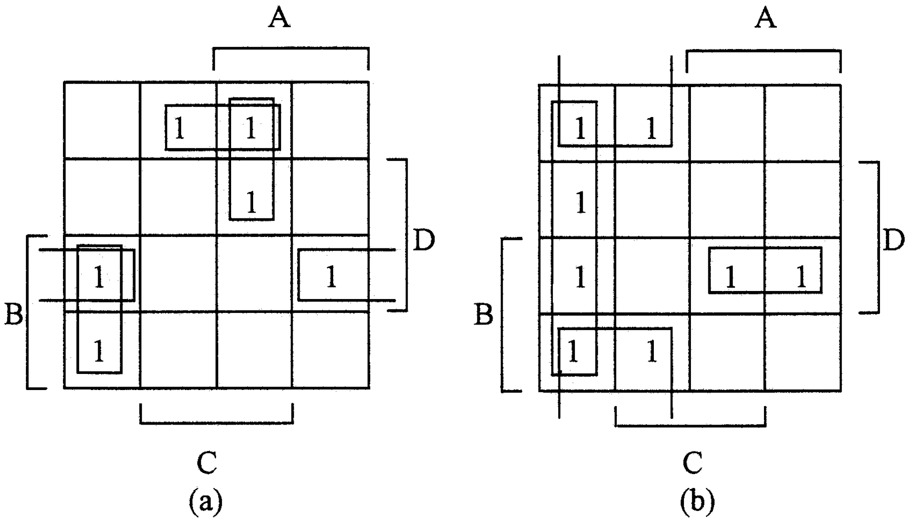


FIG. 13. The *CKM* representing the pair of *CSSFs* (*g, h*) of Example 3, (a) map of *g,* (b) map of *h.*

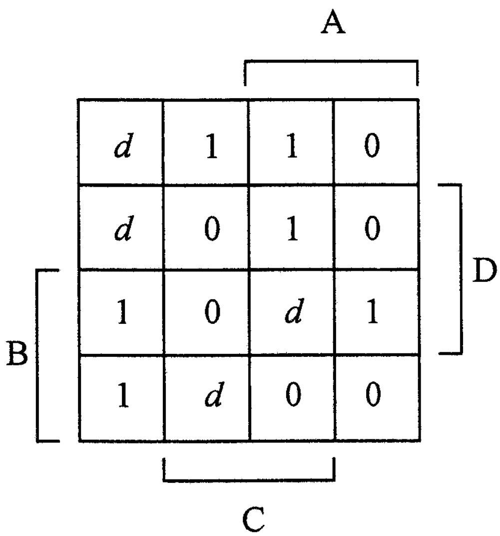


FIG. 14. The *CKM* representing the *ISSF* of $f(A,B,C,D)$ of Example 3.

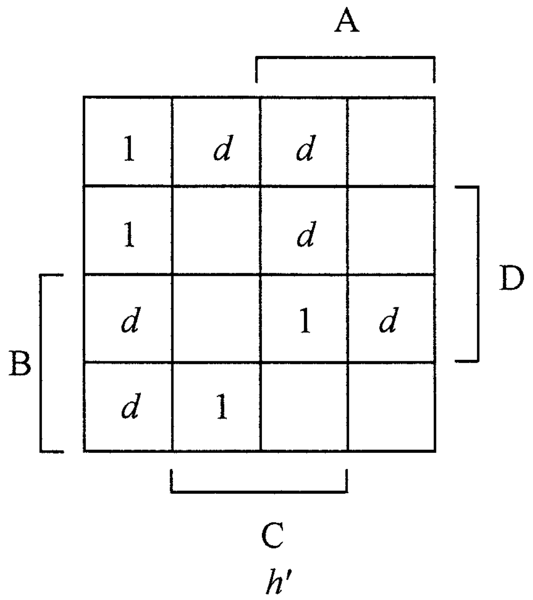


FIG. 15. The *CKM* representing the *ISSF h'* of Example 3.

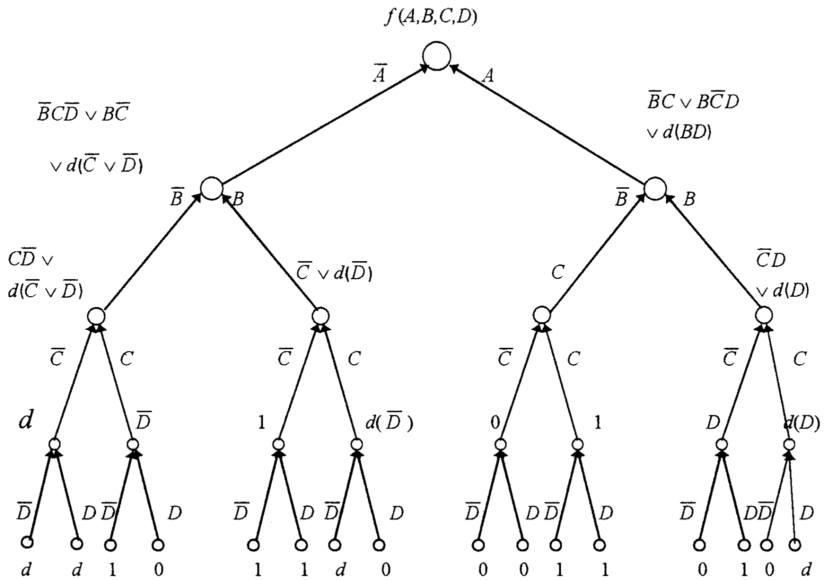


FIG. 16. An expansion tree for the function of Example 13. The arrows imply the tree interpretation as a (Mason) signal flow graph.

cessively use (1) to expand the function about its arguments *A, B, C* and *D* (the order of the arguments being selected arbitrarily). Note that the expansion tree is a complete binary tree that can also be interpreted as a Mason signal flow graph in which addition and multiplication are replaced by their logical counterparts (ORing and ANDing).

Figure 17 summarizes Figure 16 in *VEKM* form; the information at each level of the tree in Figure 16 is condensed in a corresponding *VEKM* in Figure 17. Note that as we go down the tree, the number of nodes is doubled, and hence the number of cells in the corresponding *VEKM* is doubled too.

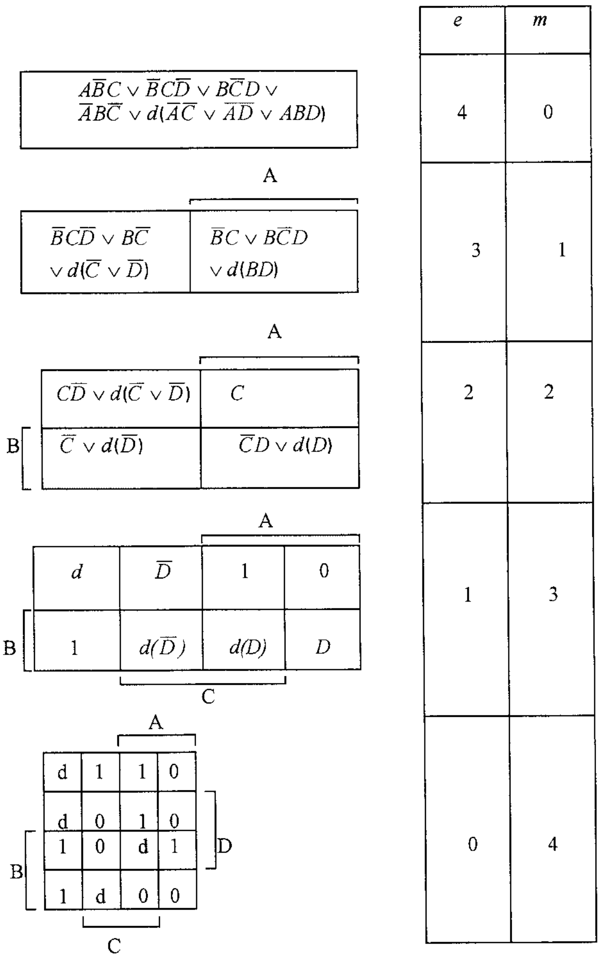


FIG. 17. A potpourri of *VEKMs* for the function of Example 3 for different numbers *e* of entered variables and *m* of map variables.

5. Duality Concepts in *VEKM* Representations

Definition 1

The dual of the n-variable *SF*

$$f(\boldsymbol{X}) = f(X_1, X_2, \ldots, X_n), \tag{20}$$

is

$$f^d(\boldsymbol{X}) = \overline{f}(\overline{X}) = \overline{f}(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n). \tag{21}$$

Note that this definition is independent of the way we represent the *SF*. A more common secondary definition can be derived for functions involving only the AND, OR and NOT operators, *e.g.,* functions represented in s-o-p or p-o-s forms. This secondary definition can be viewed as a corollary of the primary definition together with De Morgan's laws. It is given as

***Theorem 4*:** If each $\vee$, $\wedge$, 1 and 0 in a switching function involving only the AND, OR and NOT operators is replaced by $\wedge$, $\vee$, 0, and 1, respectively, then the dual of this function is obtained.

This rule for forming the dual can be summarized as follows :

$$[\, f(X_1, X_2, \ldots, X_n \;;\; 0, 1, /, \wedge\,)]^d \tag{22}$$
$$= f(X_1, X_2, \ldots, X_n \,; 1, 0, \;\wedge, /).$$

***Proof*:** Let $f(\boldsymbol{X})$ be given in an s-o-p form that does not include any of the 2 constants 0 or 1

$$f(\boldsymbol{X}) = \bigvee_i \big(\bigwedge_j \boldsymbol{X}_{ij} \big). \tag{23}$$

Take the complement (negation) of both sides of (23) and apply De Morgan's law to its right hand side to obtain

$$\overline{f}(\boldsymbol{X}) = \bigwedge_i \big(/_j \; \overline{\boldsymbol{X}}_{ij} \big). \tag{24}$$

Now complement each variable in both sides of (24) to get

$$f^d = \overline{f}(\overline{\boldsymbol{X}}) = \bigwedge_i \big(\bigvee_j \overline{\overline{\boldsymbol{X}}}_{ij} \big) = \bigwedge_i \big(\bigvee_j \boldsymbol{X}_{ij} \big). \tag{25}$$

If we now allow the expression of f to contain the constants 0 or 1 then the negation involved in going from (23) to (24) {due to De Morgan's laws} will affect them while the second negation encountered in going from (24) to (25) {due to the primary definition of duality} will apply only to variables and will not affect them. Thus, while variables remain unchanged in (22), the constants 0 and 1 are inverted.

A similar proof applies to a function expressed in a p-o-s form. The proof can be extended to general functions involving only the AND, OR and NOT operators since these can be easily reduced to ones in a p-o-s or an s-o-p form. (Q.E.D)

***Theorem 5*:** If we take the dual of both sides of a switching theorem or identity, we obtain another theorem or identity which is the dual of the original one.

Example 4

From the identity

$$X \vee Y \wedge (\overline{X} \vee \overline{Y}) = X \vee Y \ , \tag{26}$$

we get another identity,

$$X \wedge (Y \vee \overline{X} \wedge \overline{Y}) = X \wedge Y \ . \tag{27}$$

Notice that the second identity is derived in the following manner. In the left-hand side in the first identity, the priority of calculation is in the order of the last $\vee$, the $\wedge$, and the first $\vee$. In the left-hand side in the second identity, the last $\wedge$ is correspondingly to be calculated first, the $\vee$ next, and then the first $\wedge$; thus parentheses are placed as shown. The right-hand side in the second identity can be similarly obtained.

Example 5

Figure 18 gives *CKMs* for the 10 genuine (non-degenerate or non-vacuous) *SFs* of 2 variables *X* and *Y,* normally referred to as binary switching operations. The 10 maps are arranged on the vertices of a decagon such that:

(a) Contrary operations (complementary functions) are at diametrically opposite vertices.

(b) Dual operations are on the vertices of the same vertical diagonal.

Example 6

Both the *CKM* and *VEKM* are useful in obtaining the dual of a *SF*. Figures 19 and 20 give various *VEKM* representations for the function of Example 3 (Figure 17). The *VEKM* entries in Figure 19 are in p-o-s forms while those in Figure 20 are in s-o-p forms. The *CKM* entries at the bottom of Figures 19 and 20 can be obtained by a direct application of Theorem 4 to the *CKM* variables and entries at the bottom of Figure 16. Simply complement the variables and entries of the *CKM* representing the *SF* f to obtain a map representing its dual f^d. The dual *VEKM* entries can be obtained through the following theorem.

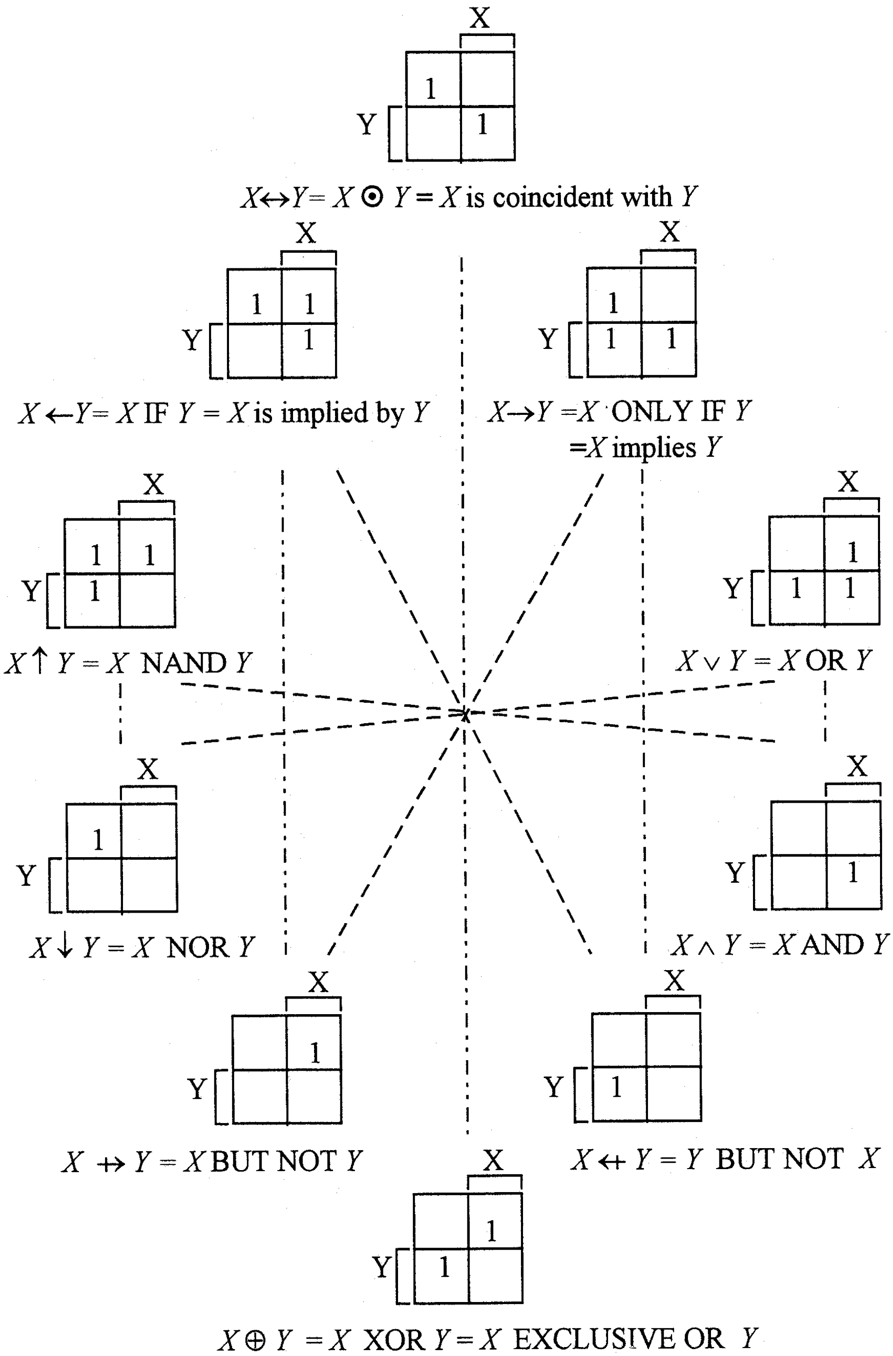


FIG. 18. The ten genuine binary operations.

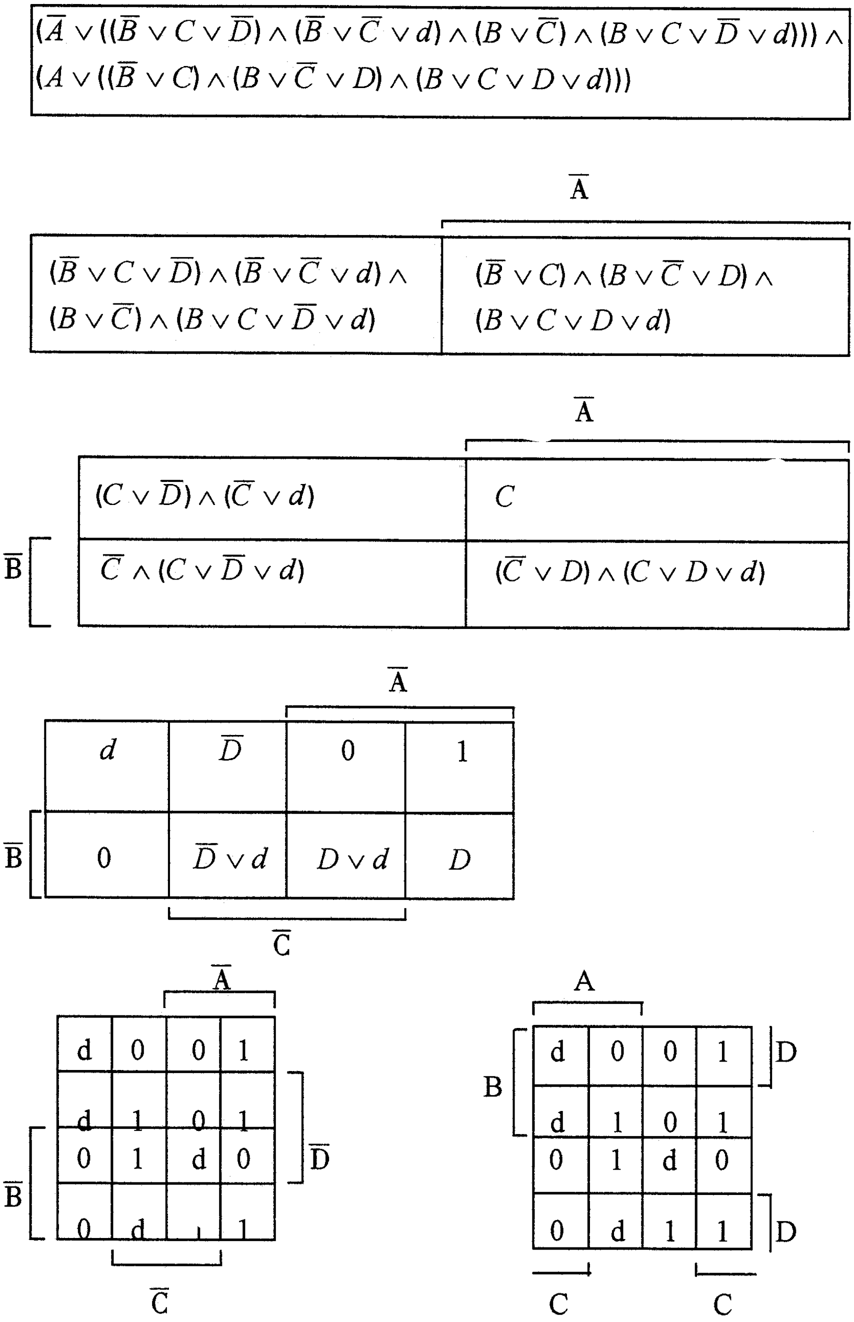


FIG. 19. A potpourri of *VEKMs* for the dual of the function in Figure 17. The *VEKM* entries are in p-o-s forms.

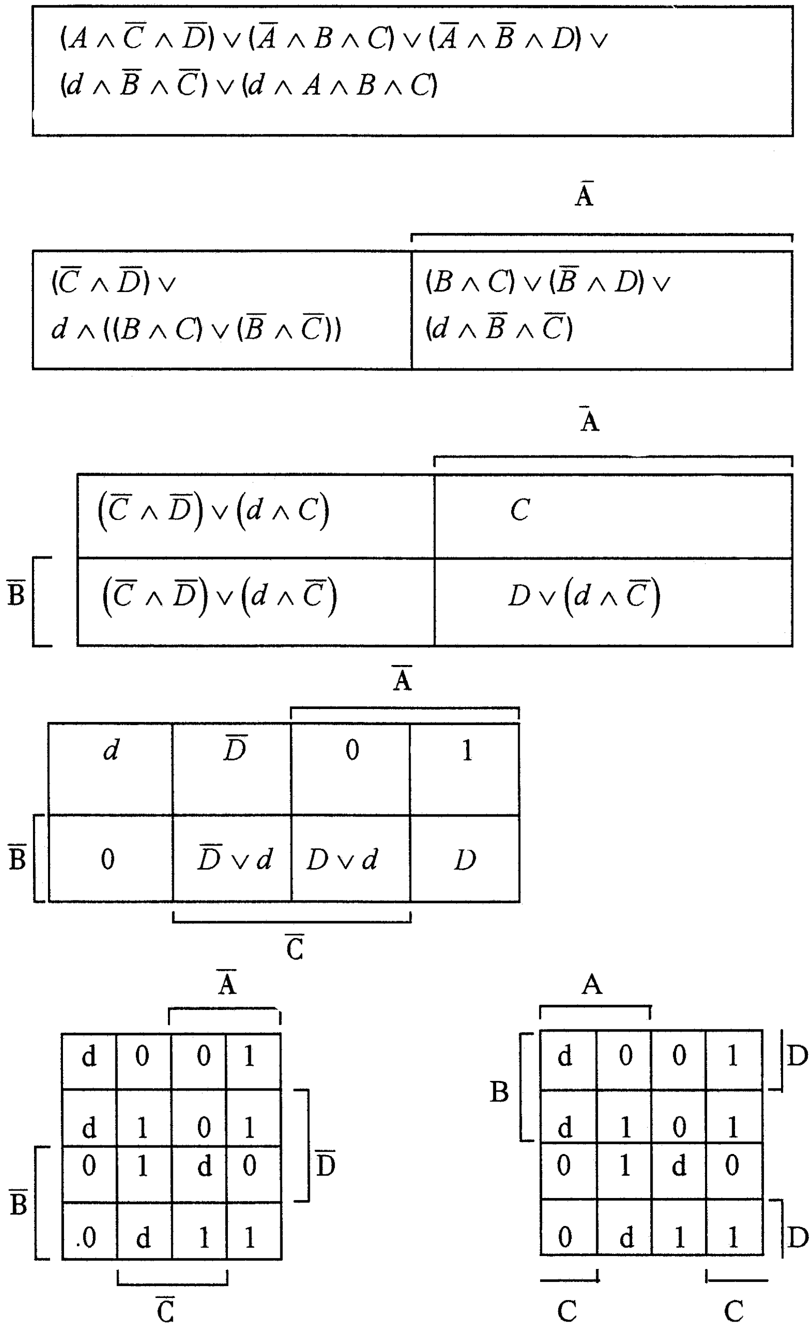


FIG. 20. A potpourri of *VEKMs* for the dual of the function in Figure 17. The *VEKM* entries are in s-o-p forms.

***Theorem 6*:** If the switching function $f(\boldsymbol{X})$ given by equation (1), namely

$$f(\boldsymbol{X}) = \overline{X}_i\, f_o\, (\boldsymbol{X} \mid X_i) \vee X_i\, f_1\, (\boldsymbol{X} \mid X_i)\,, \tag{28}$$

where the two functions $f_0\,(\boldsymbol{X} \mid X_i)$ and $f_1\,(\boldsymbol{X} \mid X_i)$ are functions of the $(n-1)$ variables obtained by excluding the variable X_i from the n variables of $\boldsymbol{X}$, and are given by

$$f_0\,(\boldsymbol{X} \mid X_i) = f(\boldsymbol{X} \mid 0_i)\,,\ f_1(\boldsymbol{X} \mid X_i) - f(\boldsymbol{X} \mid 1_i)\,, \tag{29}$$

then the dual of s is given by

$$f^d(\boldsymbol{X}) = \overline{X}_i\, f_1^d\, (\boldsymbol{X} \mid X_i) \vee X_i\, f_0^d\, (\boldsymbol{X} \mid X_i)\,. \tag{30}$$

***Proof*:** Use the definition of duality, De Morgan's law, distribution and consensus laws to obtain

$$\begin{aligned}
f^d(\boldsymbol{X}) &= \bar{f}(\overline{\boldsymbol{X}}) \\
&= \overline{(X_i f_0(\overline{\boldsymbol{X}} / \overline{X}_i) \vee \overline{X}_i f_1(\overline{\boldsymbol{X}} / X_i))} \\
&= (\overline{X}_i \vee \bar{f}_0(\overline{\boldsymbol{X}} / \overline{X}_i))\,(X_i \vee \bar{f}_1(\overline{\boldsymbol{X}} / \overline{X}_i)) \\
&= \overline{X}_i\, \bar{f}_1(\overline{\boldsymbol{X}} / \overline{X}_i) \vee X_i \bar{f}_0(\overline{\boldsymbol{X}} / \overline{X}_i) \vee \bar{f}_0(\overline{X} / \overline{X}_i)\bar{f}_1(\overline{\boldsymbol{X}} / \overline{X}_i) \\
&= \overline{X}_i \bar{f}_1(\overline{\boldsymbol{X}} / \overline{X}_i) \vee X_i \bar{f}_0(\overline{\boldsymbol{X}} / \overline{X}_i) \\
&= \overline{X}_i f_1^d(\boldsymbol{X} / X_i) \vee X_i f_0^d(\boldsymbol{X} / X_i).
\end{aligned} \qquad \text{(Q.E.D.)}$$

Theorem 6 can be expressed by the *VEKM* representation of Figure 21. In essence, Theorem 6 says that *the VEKM representation of a certain SF can lead to a VEKM are complemented and the entries are replaced by their duals.* Note that in the special case of a *CKM,* replacing the constant entries by their duals amounts to replacing them by their complements. The reader is encouraged to apply this theorem together with Theorem 4 to produce the dual *VEKM* entries in Figures 19 and 20 from the original ones in Figure 17.

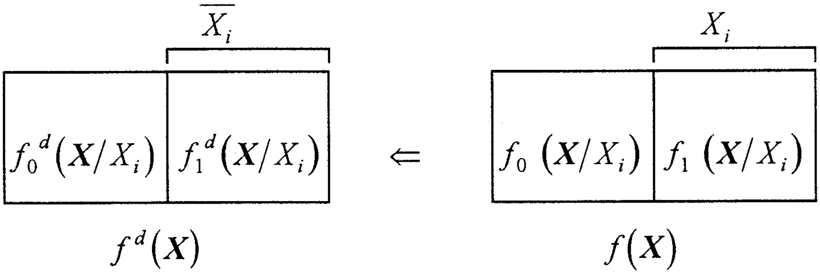


FIG. 21. A *VEKM* representation of Theorem 6.

In the following section we are going to discuss a simple *VEKM* procedure that uses sum-of-products (s-o-p) representations to obtain an *IDF* of an *ISSF*. Use of the *VEKM* to handle product of sums is also possible and is based on the product-of-sums (p-o-s) version of the Boole-Shannon expansion in[2]. As a result, a procedure that is "dual" to the present one can be used to obtain an irredundant conjunctive form (*ICF*) of an *ISSF* through constructing the disjunctive contributions of the negatively asserted prime implicates of subfunctions. The basic idea of inferring the dual procedure from the existing one is discussed in[2]. Simply stated, the dual procedure is a replica of the present one provided that any of the following words is replaced by the word trailing it in parentheses: positively asserted (negatively asserted), implicant (implicate), union (intersection), intersection (union), term (alterm), disjunction (conjunction), conjunction (disjunction), conjunctive contribution (disjunctive contribution), generalized consensus of a set of terms (dual generalized consensus of a set of alterms), *IDF* (*ICF*), minimal sum (minimal product), s-o-p (p-o-s), and the like. Note that certain terms like the words "prime", "consensus" and "subsume" are common to both dual procedures

6. A Simple VEKM Procedure for Obtaining an Irredundant Disjunctive Form for an Incompletely Specified Switching Function

As noted before, this procedure is an extension, improvement and a simplification of an earlier procedure proposed by Rushdi[2]. The outcome of the present procedure is guaranteed to be an *IDF* of the *ISSF* that is usually minimal or nearly minimal. The starting point is a *VEKM* representation of the *ISSF* such that all asserted entries are written in minimal s-o-p forms.

6.1 The Function Expression

An almost minimal s-o-p expression for the *VEKM* function f is given[2] by:

$$f = \bigcup_{r} P_r \, Co\,(P_r)\,, \tag{31}$$

where P_r is a prime implicant of one or more of the subfunctions f_i of f, i.e., it is a product that appears in at least one *VEKM* cell, and the union operator in equation (31) each P_r is called is ANDed with what is called its minimal s-o-p contribution to f, namely $Co\,(P_r)$. This contribution can be represented by a *CKM* directly deducible from the original *VEKM,* and can therefore be expressed as

$$Co\,(Pr) = \bigcup_{s} Co\,(P_r) \tag{32}$$

which represents a union of loop products $Co_s\,(P_r)$ over all prime implicant loops s in the *CKM* of $Co\,(P_r)$. Note that $Co\,(P_r)$ is a function of the map variables only while P_r itself is a function of the entered variables only. When (31) and (32) are combined, our algebraic expression for the *VEKM* function f takes the form

$$f = \bigcup_{r,s} P_r\, C_o\,(P_r)\,. \tag{33}$$

Note that the expression (33) is not necessarily an *IDF* yet.

6.2 Order of Processing the Various Contributions

The expression of $C_o\,(P_r)$ can be affected by the existence of entered implicants subsumed by or whose consensus is P_r, and also the existence of P_r can affect the contributions of asserted implicants that $C_o\,(P_r)$ covers partially. Therefore a list of all asserted selected prime implicants and another for the don't care ones must be prepared. Note should then be taken of any element of the list that subsumes or is the consensus of some other elements of it. Furthermore, the list is to be sorted in the order to be used in further processing. Generally, the P_r's involving more literals should appear earlier, so that any P_r will precede other asserted implicants, it may prove useful if P_r is placed earlier to these implicants.

6.3 Rules for Constructing and Interpreting CKMs for the (P_r) s

The *CKMs* representing the various $C_o\,(P_r)$ are obtained through the following heuristic rules. These rules are to be explained further in terms of several illustrative examples in section 7.

(1) In constructing a *CKM* for $C_o\,(P_r)$, any cell containing a don't care P_r is regarded a *d*-cell.

(2) If P_t is strictly subsumed by P_r, then any asserted or don't care cell for P_t has a *d* in the *CKM* for P_r.

(3) If P_r does not appear in a cell but two or more terms (whether asserted or don't care) that appear there have a consensus subsumed by P_r, then the cell is a *d*-cell for $C_o\,(P_r)$. Note that a consensus of more than two terms is obtained by repeatedly deriving the consensus of exactly two terms. This is known as the generalized consensus of the respective set of terms[12].

(4) If P_r is the generalized consensus of some P_s's where $s \in S$, then an asserted cell for P_r is considered a *d*-cell in $C_o\,(P_r)$ if for all values of $s \in S$ this cell is covered in the map of some $Co\,(P'_s)$ such that P'_s is subsumed by P_s. Otherwise, that asserted P_r cell is a 1-cell for $C_o\, P_r$.

(5) If P_r is the generalized consensus of some P_i's then if an asserted cell for P_r is covered in most of the *CKMs* of the $Co\,(P_i)$s, it is worth attempting to make this particular cell a *d*-cell in $C_o\,(P_r)$ and a 1-cell for the rest of the $C_o\,(P_i)$s, *i.e.,* for those $C_o\,(P_i)$s in which the cell is not covered. As an extreme case, the situation of rule (4) above arises if a cell is covered in all the *CKMs* of the $C_o\,(P_i)$s for then this cell is a *d*-cell and not a 1-cell for $C_o\,(P_r)$.

(6) If a loop *s* obtained in $C_o\,(P_r)$ is found to have all it cells as asserted or don't care cells for another product P_{rs} strictly subsumed by P_r then the loop product, to be labelled $Co_s\,(P_r)$, should be ANDed with P_{rs} and not P_r, an action to be denoted herein as an enlargement action. Such an enlargement is nec-

essary to ensure that the resulting product is a prime implicant of the *VEKM* function. In fact, while the loop s is an almost minimal contribution of P_r, it is a minimal contribution of P_{rs} , provided P_{rs} has as few literals as possible, *i.e.,* provided P_{rs} does not strictly subsume another product that appears at least in don't care form in all cells of the loop s. Formally this rule means that (33) is finally replaced by the *IDF* form:

$$f = \bigcup_{r,s} P_{rs} \, Co_s \, (P_r) \,. \tag{34}$$

In (34), P_{rs} is simply intended to mean P_r whenever no enlargement is incurred. In drawing loops in the map of $C_o \, (P_r)$, it is worth attempting to draw a smaller loop in which the above enlargement situation arises than to draw a larger loop in which it does not arise.

7. Illustrative Examples

Example 7

This is a partial example intended to demonstrate the application of rules 1-4 only. For the *VEKM* of a 7-variable *ISSF* in Figure 22(a), we focus our attention now on the construction of a *CKM* in Figure 22(b) for the contribution of one particular asserted implicant, namely X_5X_6. This *CKM* has a *d*-cell when

a) a cell contains $d(X_5X_6)$ (rule 1).

b) a cell contains either an asserted or a don't care version of each of the products X_5, X_6 and 1, since each of them strictly subsumes X_5X_6 (rule 2).

c) a cell contains asserted or don't care versions of both of the terms X_6X_7 and $X_5\overline{X}_7$ whose consensus is X_5X_6 which subsumes itself (rule 3).

The *CKM* has two 1-cells in which X_5X_6 appears asserted. In these 2 cells X_5X_6 is the only entered product, so we have no worry that any of them may be covered in contribution maps of other implicants (rule 4).

Our explicitly stated rules produced 10 *d*'s in the map $Co \, (X_5X_6)$ resulting in a coverage of 2 single-literal loops for it. If some of these *d*'s are missing the number of literals for each loop may increase up to four.

Example 8

This is again a partial example whose sole purpose is a demonstration of the enlargement rule (rule 6). The *VEKM* for the 6-variable *ISSF* in Figure 23(a) is used to construct a *CKM* in Figure 23(b) for the contribution of the sole entered asserted implicant, namely X_5X_6. Four prime implicants loops are drawn, three of which turn out to be contributions of the following terms respectively: 1, X_5 and X_6, each of which is strictly subsumed by the original term X_5X_6. The minimal sum of the original function f is

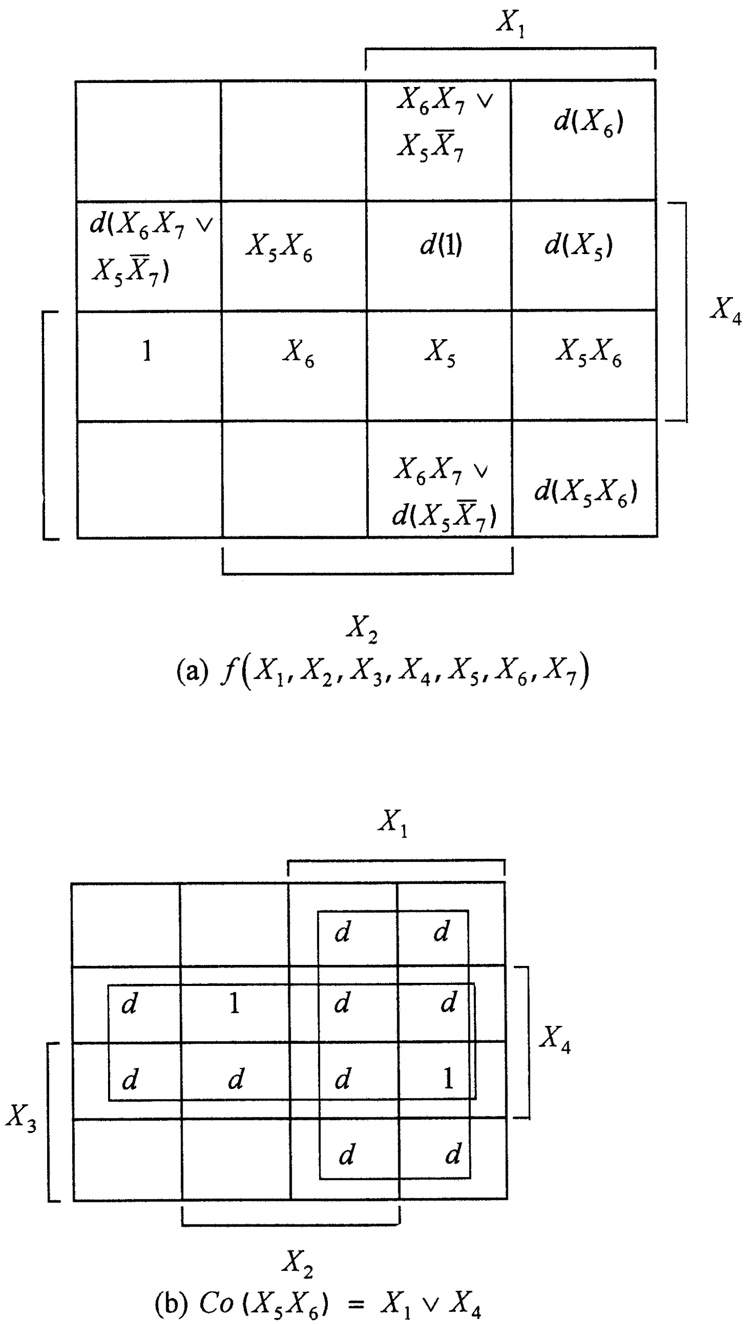


FIG. 22. The *VEKM* for the 7-variable *ISSF* in (a) is used to construct in (b) a *CKM* for the construction of one of its asserted entered implicants.

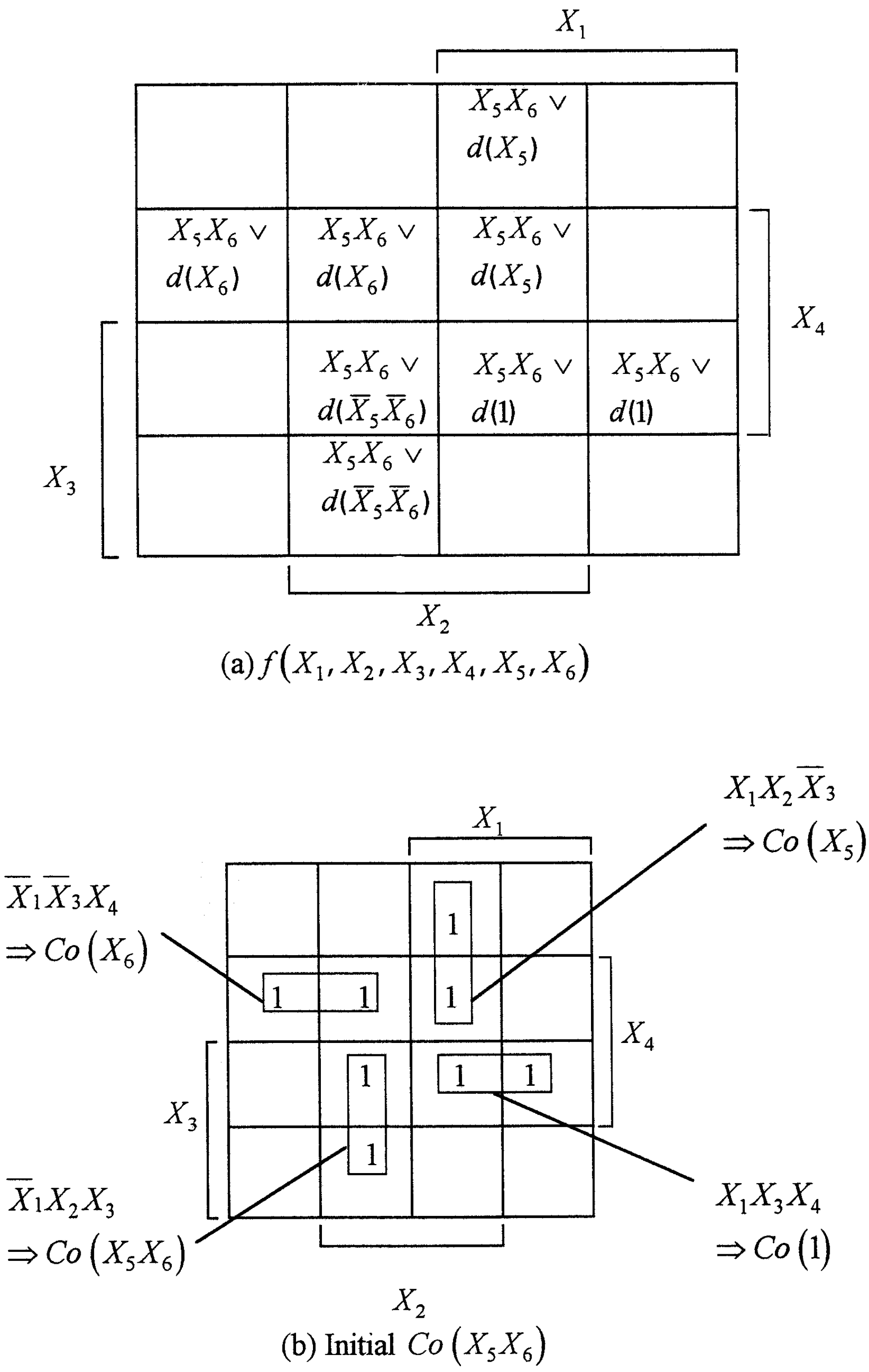


FIG. 23. Demonstration of the enlargement rule.

$$f = X_1X_3X_4 \vee X_1X_2\overline{X}_3X_5 \vee \overline{X}_1\overline{X}_3X_4X_6 \vee \overline{X}_1X_2X_3X_5X_6 . \tag{35}$$

If the enlargement rule is not used, f is given by the non-minimal form

$$f = X_1X_3X_4X_5X_6 \vee X_1X_2\overline{X}_3X_5X_6 \vee \overline{X}_1\overline{X}_3X_4X_5X_6 \vee \overline{X}_1X_2X_3X_5X_6 . \tag{35a}$$

Example 9 *(Example 2 revisited)*

To obtain an *IDF* for the function of Example 2, we redraw its *VEKM* representation in Figure 5 with its subfunctions rewritten in minimal s-o-p forms (see Figure 24). This is an example of a completely specified switching function (*CSSF*), and hence only rules 2, 3, 4 and possibly 5 are applicable in this case.

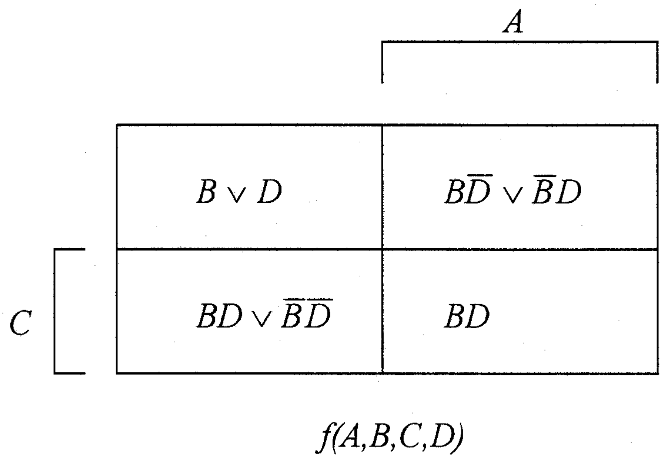


FIG. 24. The function of Figure 5 with its subfunctions rewritten in minimal s-o-p forms.

The prime implicants of the subfunctions that appear in the cells of the *VEKM* of Figure 24 are $\overline{B}D$, $B\overline{D}$, $\overline{B}\overline{D}$, BD, D and B whose subsumption relations are shown in Figure 25. Beside these subsumptions, we note that B is the consensus of $B\ \overline{D}$ and BD while D is the consensus of BD and $\overline{B}\ D$. None of the

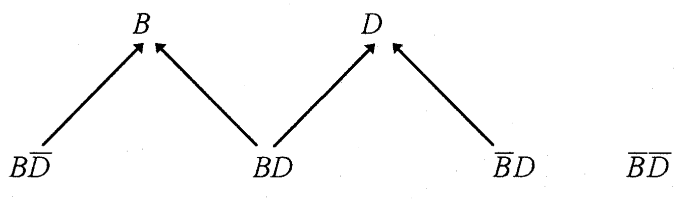


FIG. 25. The relation between the subsumed terms (top row) and the subsuming terms (bottom row) of the prime implicants of the subfunctions of the *CSSF* in Figure 24.

2-literal terms $\overline{B}D$, $B\overline{D}$, $\overline{B}\,\overline{D}$ and BD is subsumed by or is the consensus of any of the other terms. The contributions of the entered implicants are obtained by the *CKMs* shown in Figure 26. Table 1 will help the reader understand how these *CKMs* are constructed. The final expression for the function f is:

$$f = \overline{C}B\overline{D} \vee CBD \vee \overline{C}\,\overline{B}D \vee \overline{A}C\overline{B}\,\overline{D} \vee \overline{A}\,\overline{C}B\,, \tag{36}$$

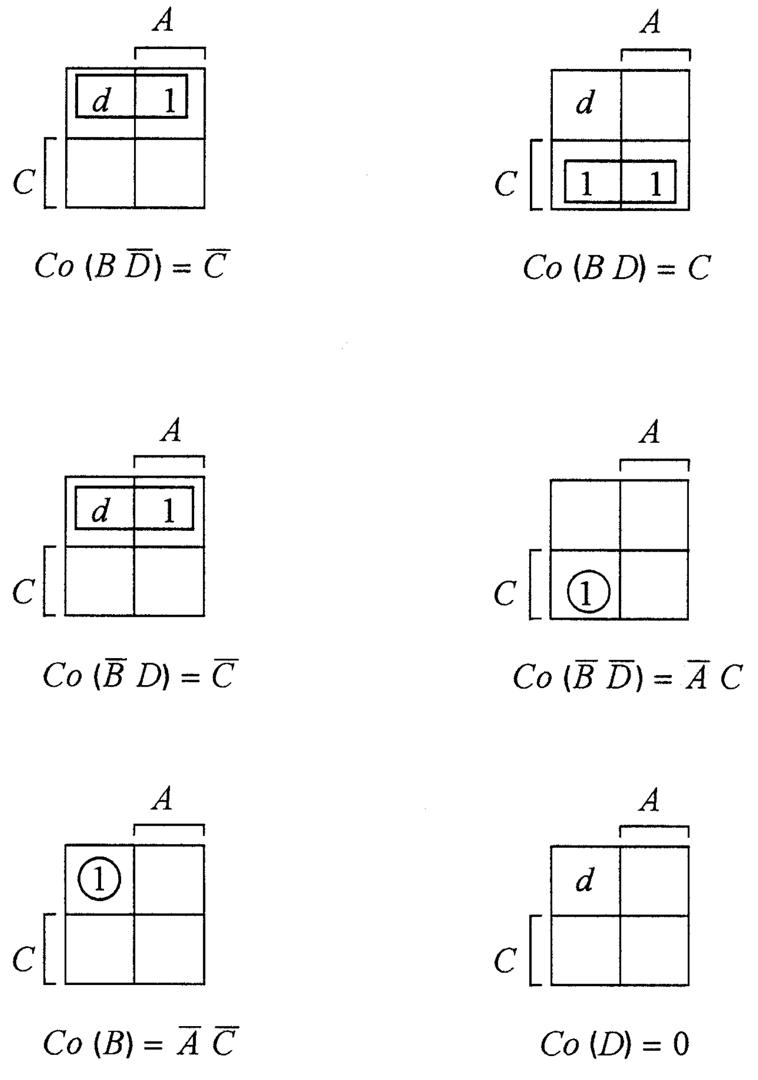


FIG. 26. The *CKMs* representing the contributions of the prime implicants of the subfunctions given in Figure 24.

which is one of three *IDFs* that the function f has. Note that if rule 4 is overlooked, *Co* (*D*) may be taken as AC rather than 0 and consequently expression (36) ceases to be an *IDF* as it is augmented by the redundant term $\bar{A}\bar{C}D$.

TABLE 1. Remarks on how the *CKMs* of Figure 26 are constructed.

CKM for	Remarks
$B\bar{D}$	The $\bar{A}\bar{C}$ cell is a *d*-cell since the term B that is strictly subsumed by $B\bar{D}$ appears there (rule 2). The $A\bar{C}$ cell is a 1-cell (rule 4) since $B\bar{D}$ appears there, and it is neither subsumed by nor is the consensus of any of the other terms.
BD	The $\bar{A}\bar{C}$ cell is a *d*-cell since the term BD strictly subsumes either of the terms B and D which appear there (rule 2). Both $\bar{A}C$ and AC are 1-cells since the 2-literal term BD appears there (rule 4).
$\bar{B}D$	The $\bar{A}\bar{C}$ cell is a *d*-cell since the term D that is strictly subsumed by $\bar{B}D$ appears there (rule 2). The $A\bar{C}$ cell is a 1-cell (rule 4) since the 2-literal term $\bar{B}D$ appears there.
$\bar{B}\bar{D}$	The $\bar{A}C$ cell is a 1-cell (rule 4) since the 2-literal term $\bar{B}D$ appears there.
B	The $\bar{A}\bar{C}$ cell is a 1-cell since B appears there. There are 2 terms $B\bar{D}$ and BD or D whose consensus is B (see Figure 25). However, though the $\bar{A}\bar{C}$ cell is covered in *Co* ($B\bar{D}$) it is not covered in *Co* (BD) or *Co* (D) (rule 4).
D	The $\bar{A}\bar{C}$ is a *d*-cell since D appears there and there are 2 terms $\bar{B}D$ and B (Figure 25) whose consensus is D, with the $\bar{A}\bar{C}$ cell covered with both *Co* (B) and *Co* ($\bar{B}D$).

Another *IDF* of f is easily obtained if the order of processing the entered implicants B and D is reversed, thereby modifying their contributions to *Co* (*D*) = $\bar{A}\ \bar{C}$ and *Co* (*B*) = 0. This *IDF* is

$$f = \bar{C}B\bar{D} \vee CDB \vee \bar{C}\bar{B}D \vee \bar{A}C\bar{B}\bar{D} \vee \bar{A}\bar{C}D . \tag{37}$$

A third *IDF* can be obtained through the use of rule 5 as shown in the *CKMs* of Figure 27. Since D is the consensus of BD and $\bar{B}D$ and the cell $\bar{A}\bar{C}$ is covered in *Co* (*BD*), we explore the possibility of making this cell a 1-cell for *Co* (*BD*) as shown in Figure 27. This turns out to be a useful attempt since it does not only make $\bar{A}\ \bar{C}$ a *d*-cell in *Co* (*D*) but it makes that cell covered in both *Co* ($B\bar{D}$) and *Co* (*BD*), and therefore it allows us to make this cell a *d*-cell in *Co* (*B*), since B is the consensus of $B\bar{D}$ and BD. Finally f is expressed by the *IDF*

$$f = \bar{C}B\bar{D} \vee CBD \vee \bar{C}\bar{B}D \vee \bar{A}C\bar{B}\bar{D} \vee \bar{A}BD \tag{38}$$

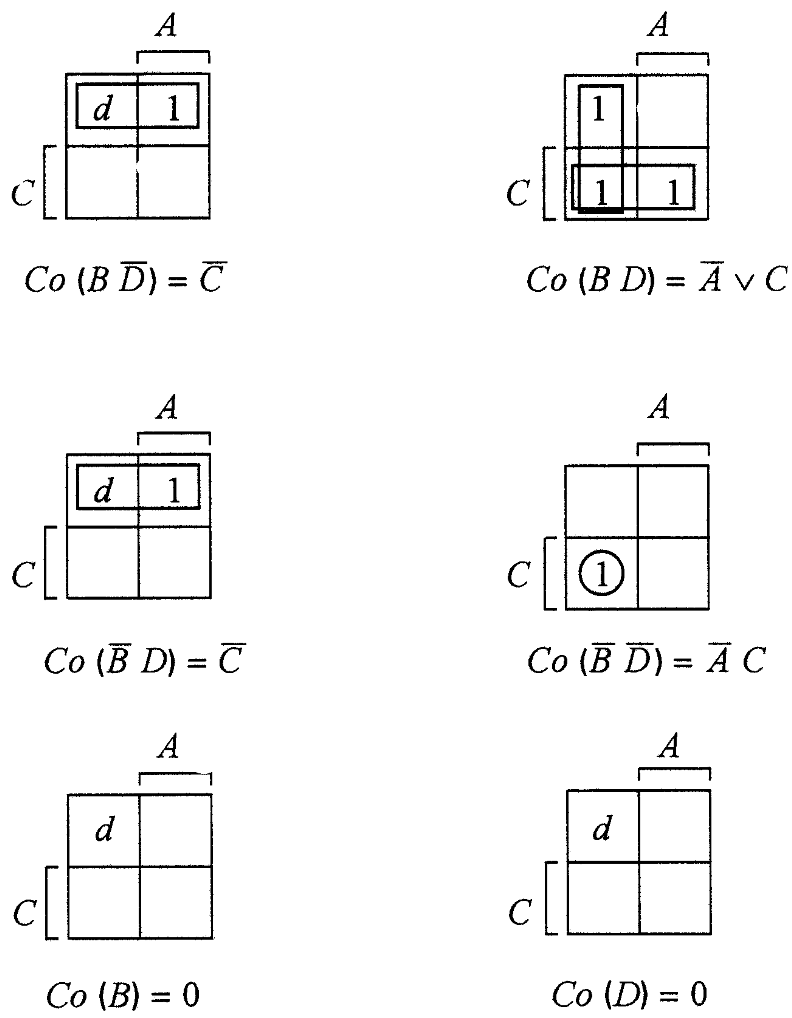


FIG. 27. The third solution for the *CSSF* in Figure 24.

It can be shown that the formulas that appear in (36)-(38) are the only *IDFs* of *f*. All three of them are minimal.

Example 10 *(Example 3 revisited)*

Consider the function $f\,(A,B,C,D)$ previously discussed in Example 3. Starting by the *VEKM* representation of this function in Figure 17 for which $e = m = 2$, we note that the prime implicants of the subfunctions that appear in the cells of that *VEKM* are asserted $\overline{C}D$, $C\overline{D}$, $\overline{C}$ and C together with don't care D, $\overline{D}$, and $\overline{C}$.

The contributions of the asserted implicants are obtained by the *CKMs* shown in Figure 28 which are constructed in accordance with rules 1-6. In particular we note the following. Since in the 2 cells of the loop $\bar{A}$ covering *Co* ($C\,\bar{D}$), the product $C\bar{D}$ can be unioned with $\bar{C}\bar{D}$ (which is present as a part of $\bar{C} = \bar{C}\bar{D} \vee \bar{C}D$), then the contributing product $C\bar{D}$ replaced by $\bar{D}$ (Rule 6). This replacement is useful in simplifying the map for *Co* ($\bar{C}$), since now its cell $\bar{A}B$ is entered by a *d* rather than 1 in accordance with Rule 4; the $\bar{C}$ in that cell is totally covered in the maps of *Co* ($\bar{C}D$) and *Co* ($\bar{D}$). The final expression for the function f is:

$$f = B\bar{C}D \vee \bar{A}\bar{D} \vee A\bar{B}C. \tag{39}$$

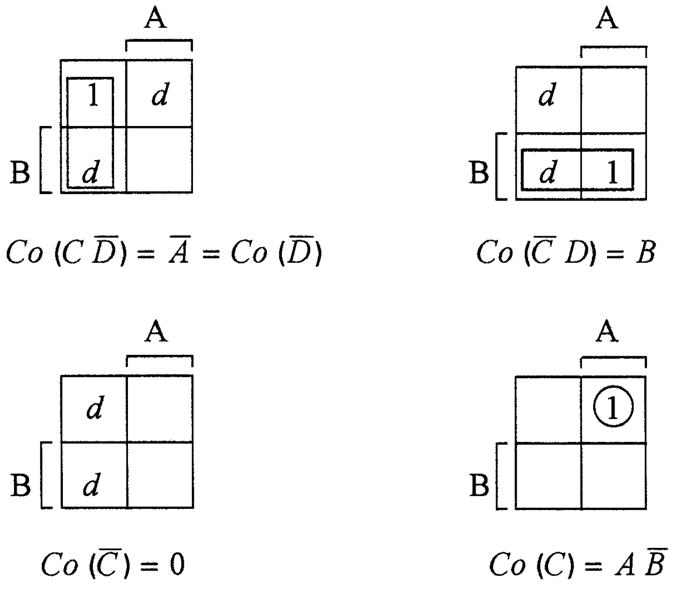


FIG. 28. The *CKMs* representing the contributions of the asserted implicants of the subfunctions given by the *VEKM* in Figure 17 for $e = m = 2$.

This minimal s-o-p expression for f is also obtained when a number of map variables m other than 2 is used. For example, we can use the *VEKM* in Figure 17 for which $e = 1$ and $m = 3$ as our starting point, thereby producing the *CKMs* in Figure 29 for the contributions of the asserted implicants D, $\bar{D}$ and 1. These asserted implicants are related by a consensus relation ($D \vee \bar{D} = 1$) with both D and $\bar{D}$ strictly subsuming 1. Since the $\bar{A}B\bar{C}$ cell is an asserted cell for the implicant 1, and since it must be covered partially by a loop in the map for *Co* ($\bar{D}$), we have given a priority to covering the same cell in the *Co* (D) map, *i.e.,* we

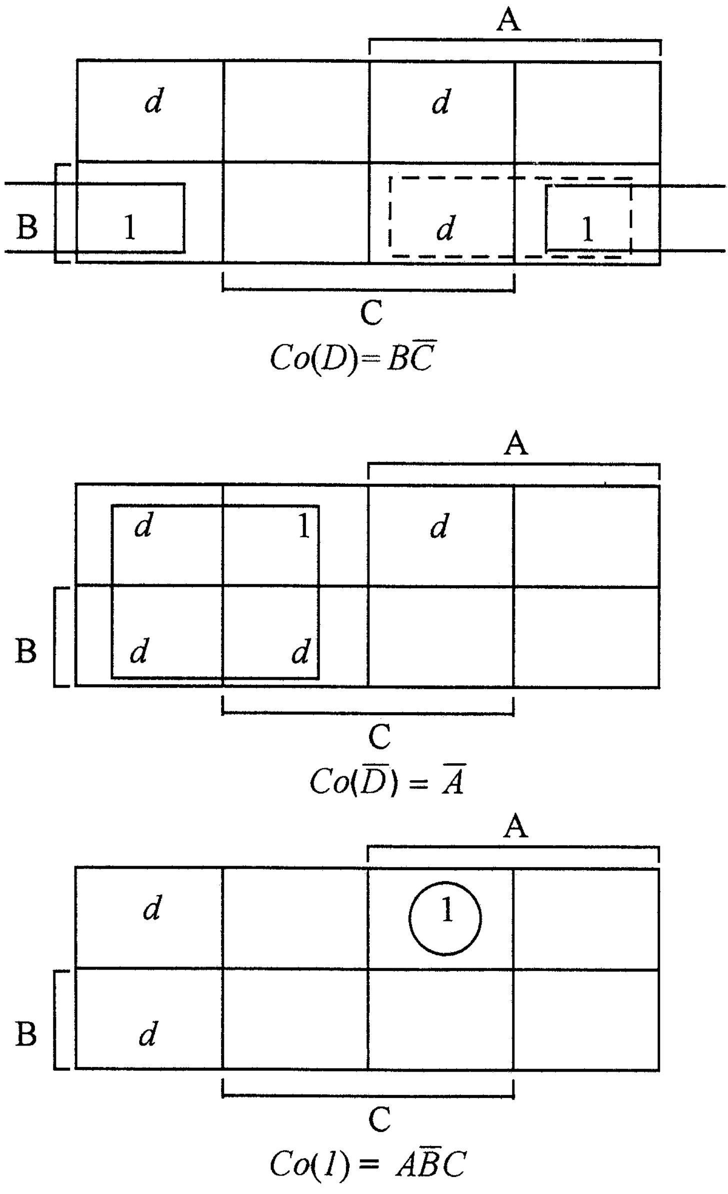


FIG. 29. The *CKMs* representing the contributions of the asserted implicants of the subfunctions given by the *VEKM* in Figure 17 for $e = 1$ and $m = 3$.

preferred to draw the loop $B\overline{C}$ rather than the loop AB(shown dotted). This allows us (Rule 5) to consider the $\overline{A}B\overline{C}$ cell in *Co* (1) map a *d*-cell rather than a 1-cell, and simultaneously consider the same cell in the *Co* (D) map a 1-cell rather than a *d*-cell which is immaterial since it is covered anyhow. Finally we obtain the same expression for the function as in (39). In passing, we note that the *VEKM* in the present case with $e = 1$ is the algebraically simplest genuine *VEKM*; it is next in algebraic simplicity only to a *CKM* (which is a *VEKM* with $e = 0$).

We now turn our attention to the *VEKM* with $e = 3 = n - 1$. This is the algebraically most difficult genuine *VEKM,* being algebraically simpler than only a purely algebraic expression (which is a *VEKM* with $e = 4 = n$). The prime implicants of the 2 subfunctions in the *VEKM* and their subsumption relations are shown in Figure 30. The contributions of the asserted implicants are obtained by the *CKMs* shown in Figure 31 which are constructed in accordance with rules 1-6. In particular we note the following. In covering *Co* ($\overline{B}D\overline{C}$) we are tempted to use the larger loop of 1 (shown dotted) but we prefer (Rule 5) to use the smaller loop of $\overline{A}$ (shown solid) with the minor gain that this latter loop is now considered a contribution of $\overline{D}$ and not of its subsuming term $\overline{B}C\overline{D}$. As a bonus, we note that the cell $\overline{A}$ in the *Co* ($B\overline{C}$) map is now totally covered since it has been covered in *Co* ($\overline{D}$) and *Co* ($B\overline{C}D$) where $B\overline{C}$ is the consensus of $\overline{D}$ and $B\overline{C}D$. Hence the $\overline{A}$ cell in the *Co* ($B\overline{C}$) map is not a 1-cell but a *d*-cell and *Co* ($B\overline{C}$) turns out to be 0. Finally we obtain the same expression for the function as in (39). The corresponding expression that would have been obtained by the procedure in[2] is the non-minimal form

$$f = B\overline{C}D \vee \overline{B}C\overline{D} \vee \overline{A}B\overline{C} \vee A\overline{B}C. \tag{39a}$$

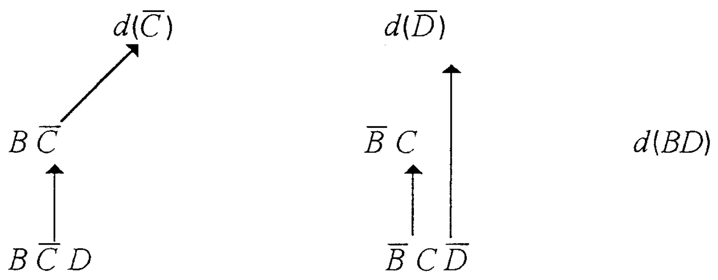


FIG. 30. Subsumption relations among the various implicants of the subfunctions in the *VEKM* of Figure 17 with $e = 3$ and $m = 1$.

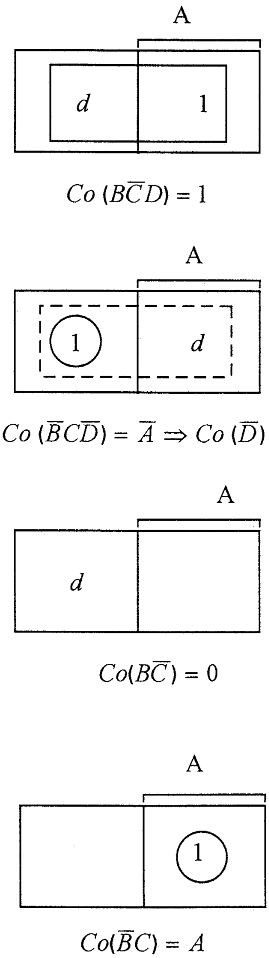


FIG. 31. The *CKMs* representing the contributions of the asserted implicants of the subfunctions given by the *VEKM* in Figure 17 for $e = 3$ and $m = 1$.

Example 11

The purpose of this example is to briefly demonstrate the dual *VEKM* procedure, *i.e.,* the one that obtains an *ICF* for a given *ISSF*. Figure32 depicts a *VEKM* for a 6 variable *ISSF* whose entries are in p-o-s forms. The (negatively) asserted entries are already in minimal p-o-s forms. Figure 33 shows the dual contributions for all (negatively) asserted entered implicates, namely $(D \vee F)$, $(D \vee E)$, D, F and 0. A final *ICF* of f is:

$$f = (D \vee F)\,(A \vee D \vee E)\,(\overline{B} \vee D)\,(C \vee D)\,(\overline{B} \vee F)\,(A \vee B \vee C)\,. \tag{40}$$

			A	A
	0	$D\,d(E\,F)$	$D\,F\,d(E)$	$D\,d(F)$
C	$(D \vee E)\,(D \vee F)$	$F\,d(0)$	$D\,d(E\,F)$	$d(F)$
		B	B	

FIG. 32. A *VEKM* for a 6-variable *ISSF* with entries in p-o-s forms.

			A	A
	d	d	d	d
C	0	d	d	d
		B	B	

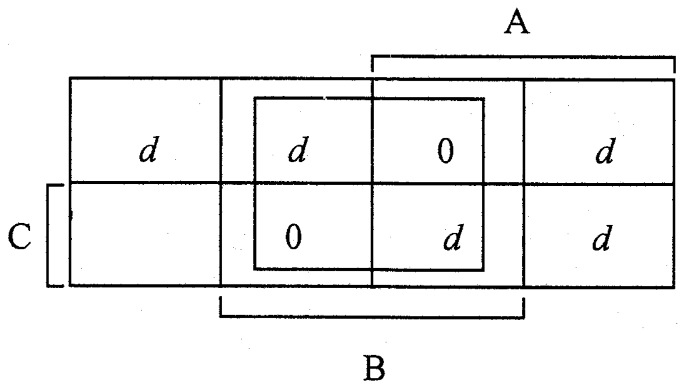
(a) $Co'(D \vee F) = 0$

			A	A
	d	d	d	d
C	0	d	d	
		B	B	

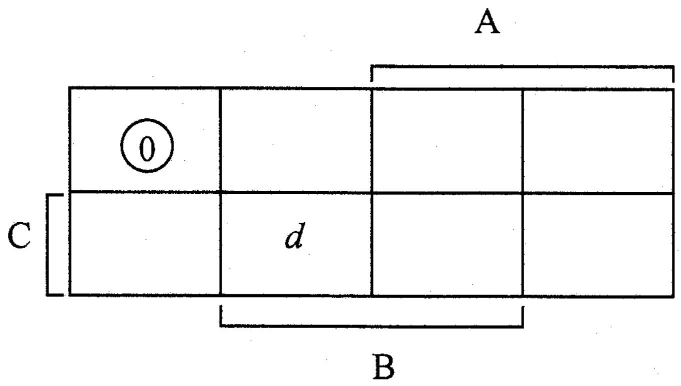
(b) $Co'(D \vee E) = A$

			A	A
	d	0	0	0
C		d	0	
		B	B	

(c) $Co'(D) = \overline{B}\,C$



(d) $Co'(F) = \overline{B}$



(e) $Co'(0) = A \vee B \vee C$

FIG. 33. The various dual contributions for the entered asserted implicates of the function in Figure 32.

8. Conclusions

This paper has given a thorough exposition of the essential features and properties of the variable-entered Karnaugh map (*VEKM*). This exposition has culminated in the presentation of two new *VEKM* procedures for obtaining an irredundant disjunctive form (*IDF*) and an irredundant conjunctive form (*ICF*) of an incompletely specified switching function (*ISSF*). Both procedures are simpler than and superior to similar procedures already existing in the literature. This fact is attested to by several examples for which the present procedures have reached exact minimality, whereas earlier procedures would have fallen short of achieving such an objective. In general, the results of the present procedures are expected always to be at least nearly minimal, with exact minimality being achieved more often than not.

The present two procedures are simple and obtain the contributions of the entered terms or alterms in one step or in one pass. By contrast, more involved 2-pass procedures can be devised such that subsumption and consensus interactions among entered terms or alterms can be fruitfully utilized to update their initial contributions and obtain some final contributions in as many different ways as possible. The reward that may warrant such an effort is that all *IDFs* and all *ICFs* of an *ISSF* may be obtained. However, to achieve this purpose, a thorough understanding of the somewhat difficult concept of the "generalized consensus" is required. Again, the *VEKM* can be used to advantage, as it allows a direct illustration of that concept as well as practical means for evaluating generalized consensi.

Two-step procedures can be also introduced along a different direction, namely, the classical direction of (a) finding all prime implicants or implicates of the *ISSF* and then (b) using these for obtaining a minimal cover for it. In fact, the *VEKM* can be easily and efficiently be utilized for implementing both steps (a) and (b) above. Step (a) is very important in its own right apart from its being a part of the minimization process. That is because it is an essential step in both hazard analysis and solution of Boolean equations. This step can have many *VEKM* implementations wherein the complete sum of the *ISSF* $g \vee d(h)$ is obtained as that of the associated *CSSF* $g \vee h$. For example, the *VEKM* can be used for obtaining any p-o-s expression of a *CSSF* which can be multiplied out to produce the complete sum after deletion of any absorbably terms. A technique of *VEKM* folding, that starts with a *VEKM* of complete-sum entries can end up with the required complete sum absorptions are implemented after each folding. The previous two techniques are *VEKM* versions of already existing algorithms. A third possible *VEKM* technique has no non-*VEKM* counterpart and minimizes the number of absorptions required. It starts with a *VEKM* of complete-sum entries and obtains the complete contributions of such entries and those of any new entries in the *VEKMs* representing the meet derivatives of the original *CSSF.* Dual versions of these various techniques can be used to derive the complete product rather than the complete sum. The *VEKM* is not less useful in implementing step (b) above, *i.e.,* in using the complete sum/product as a starting point for deriving a minimal sum/product of an *ISSF.* Several approximate or exact *VEKM* or *VEKM*-related methods for this step can be developed. We hope to write some sequels of the present paper to present the ideas above in detail as well as to further develop some of the existing and new *VEKM* applications referred to in the introduction.

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إجراءان لخريطة كارنوه متغيرة المحتويات ذوا تحسينات إضافية يستخدمان للحصول على الصيغ غير الوافرة لدالة تبديلية غير كاملة التحديد

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المستخلص . تتميز طريقة خريطة كارنوه من بين مجموعة الطرائق المتاحة للتصغير الأعظمي التقليدي للدوال التبديلية بأنها طريقة يدوية سريعة تزود مستعملها ببصيرة تصويرية . وثمة شكل متطور من هذه الخريطة هو خريطة كارنوه متغيرة المحتويات (خ ك غ ح) يضاعف عدد المتغيرات التي تتعامل معها الخريطة فضلا عن أنه يسمح باستخدامها مع أنواع الجبر البولاني الكبيرة . تقدم ورقة البحث هذه عرضا توضيحيا مبتكرا للملامح والخصائص الأساسية لهذا النوع الأخير من الخريطة معظمها ينشر لأول مرة ، كما تقدم إجراءً لهذه الخريطة يتسم بالبساطة ويتمتع بتحسينات إضافية يستخدم للحصول على إحدى الصيغ غير الوافرة لمجموع المضروبات لدالة تبديلية غير كاملة التحديد . ويتم استغلال مفاهيم الازدواجية لتحويل هذا الإجراء إلى إجراء مزاوج يستخدم للحصول على صيغة غير وافرة لمضروب المجموعات لمثل هذه الدالة. ويختلف هذان الإجراءان عن الإجراءات السابقة من ناحيتين : الأولى أن الإجراءين الحاليين هما إجراءان أكثر تطورا مجهزان بمجموعة من القواعد المنصوص عليها صراحة تتصف بأنها أكثر وضوحا رغم كونها أفضل قدرة من نظيراتها في الإجراءات السابقة ، والثانية أن الإجراءين الحاليين أكثر دقة في معالجتهما لإسهامات مضروب أو مجموع حروف أحادية مدخل في الخريطة ، ومن ثم فإن فرصتهما أرجح في أن يحصرا التفصيلات الفرعية الصغيرة في البنية الذاتية للدالة التبديلية غير كاملة التحديد التي هي محل الاعتبار . ولهذا السبب فإن

الإجراءين الحاليين ، إن جرى اتباعهما بدقة تامة ، لهما فرصة أرجح لبلوغ التصغير الأعظمي بلوغا تاما ، وحتى إن لم يتحقق ذلك فإن من المضمون دائما أن تكون التعبيرات الناشئة عنهما مقاربة للصورة الصغرى . تتضمن الورقة أمثلة توضيحية عديدة للكشف عن الملامح والخصائص الأساسية للخريطة ولتفسير قواعد وخطوات الإجراءين الجديدين .